INVERSE THERMAL DESIGN OF THERMO-MAGNETICALLY DRIVEN BOUSSINESQ FLOWS

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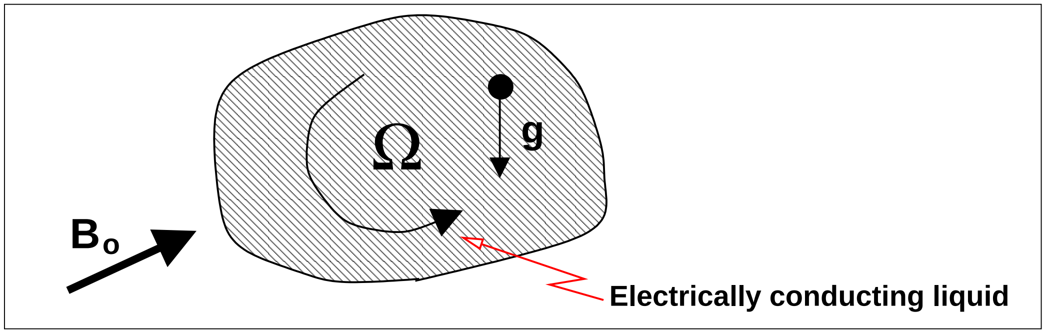
OUTLINE OF THE PRESENTATION

- Review of previous work on inverse fluid flow problems
- Interaction of an externally applied magnetic field with fluid flow
- Mathematical definition of the inverse magneto-convection problem
- Adjoint method formulation for the solution of the inverse problem posed as a functional optimization problem
- Highlights of the FEM techniques used for the solution of the direct, adjoint and sensitivity problems
- H^1 regularization to handle large errors in input data
- Numerical results
- Extension of the developed method to address the design of directional solidification processes
- Summary

PREVIOUS WORK ON INVERSE FLUID FLOW PROBLEMS

- **Moutsoglou (1989)**
  - steady state inverse free convection problem
- **Gunzburger and Lee (1994)**
  - addressed control of temperature peaks along bounding surfaces of containers
  - flow field was decoupled from the heat transfer analysis
- **Berggren, Glowinski and Lions (1996)**
  - controllability issues of flow related models (e.g. viscous Burgers equation)
- **Zabaras and Yang (1997)**
  - inverse natural convection problems
  - applications to design of directional solidification processes
- **Berggren (1998)**
  - vorticity control problem

INTERACTION OF AN APPLIED STRONG MAGNETIC FIELD WITH FLUID FLOW IN A BOUNDED DOMAIN



MODEL ASSUMPTIONS

- Walls of the cavity are electric insulators
- Magnetic Reynolds number is sufficiently small that the induced magnetic field is negligible in comparison to the imposed constant magnetic field B_o

ELECTROMAGNETIC RELATIONS GOVERNING THE MHD FLOW PROBLEM

- Lorentz body force term in the Navier-Stokes equations

$$F = J x B + \underbrace{\rho_e E}$$

Neglected as ρ_e is small in liquid metals

- Ohm's law for moving medium

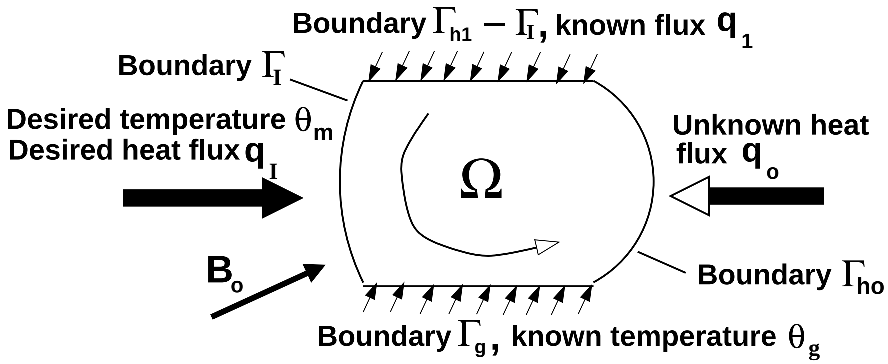
$$J = \sigma_e(- \nabla\phi + v x B) + \underbrace{\rho_e v}$$

Neglected as ρ_e is small in liquid metals

- Conservation of electric current

$$\nabla \cdot J = 0$$

MATHEMATICAL DEFINITION OF THE INVERSE MAGNETO-CONVECTION PROBLEM



Pose the inverse problem as an unconstrained spatio-temporal optimization problem

Find a (quasi-) solution $\bar{q}_o \in L_2(\Gamma_{ho} \times [0, t_{max}])$ such that:

$$J(\bar{q}_o) \leq J(q_o), \; \forall \; \bar{q}_o \in L_2(\Gamma_{ho} \times [0, t_{max}]),$$

where

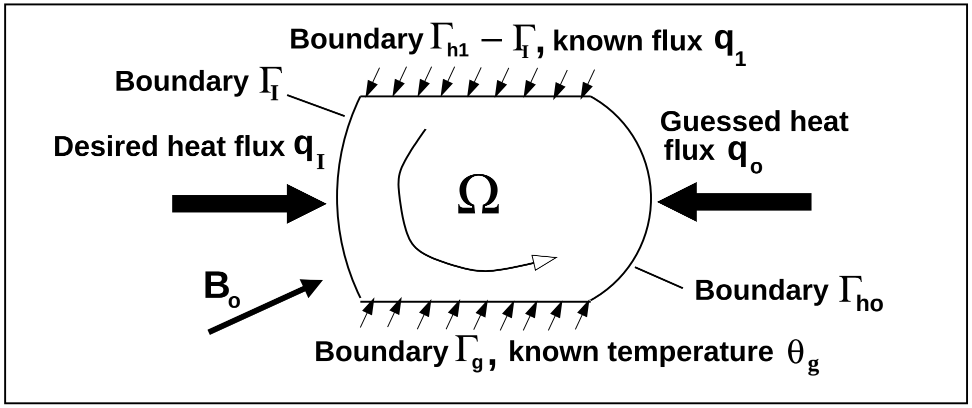
$$J(q_o) = \frac{1}{2} \left\| \theta(x,t;q_o) - \theta_m(x,t) \right\|^2_{L_2(\Gamma_I \times [0, t_{max}])}$$

$$= \frac{1}{2} \int_o^{t_{max}} \int_{\Gamma_I} [\theta(x,t;q_o) - \theta_m(x,t)]^2 \, d\Gamma \, dt$$

Measured/desired temperature

Solution of a direct magneto-convection problem
for a given guessed heat flux q_o on the boundary Γ_{ho}

WELL-POSED DIRECT MAGNETO-CONVECTION PROBLEM FOR A GIVEN HEAT FLUX q_0 ON THE BOUNDARY Γ_{ho}



MATHEMATICAL PROBLEM

- Incompressibility condition
- Navier-Stokes equations with body force terms involving Lorentz force as well as buoyancy effects
- Thermal transport equation
- Electromagnetic potential equation
- No-slip condition on all boundaries
- Electrically insulating condition on all boundaries
- Problem specific temperature/flux thermal boundary condition

CALCULATION OF THE GRADIENT OF THE OBJECTIVE FUNCTION

From the definition: $J(q_o) = \frac{1}{2} \left\| \theta(x,t;q_o) - \theta_m(x,t) \right\|^2_{L_2(\Gamma_I \times [0,\, t_{max}])}$

$$\boxed{\begin{aligned} D_{\triangle q_o} J(q_o) &\equiv (J'(q_o), \triangle q_o)_{L_2(\Gamma_{h0} \times [0,\, t_{max}])} \\ &= (\theta(x,t;q_o) - \theta_m(x,t), \Theta(x,t;q_o,\triangle q_o))_{L_2(\Gamma_I \times [0,\, t_{max}])} \end{aligned}}$$

Sensitivity temperature field $\Theta(x,t;q_o,\triangle q_o) \equiv D_{\triangle q_o}\theta(x,t;q_o)$

Sensitivity velocity field $\boldsymbol{V}(x,t;q_o,\triangle q_o) \equiv D_{\triangle q_o}\boldsymbol{v}(x,t;q_o)$

Sensitivity potential field $\Phi(x,t;q_o,\triangle q_o) \equiv D_{\triangle q_o}\phi(x,t;q_o)$

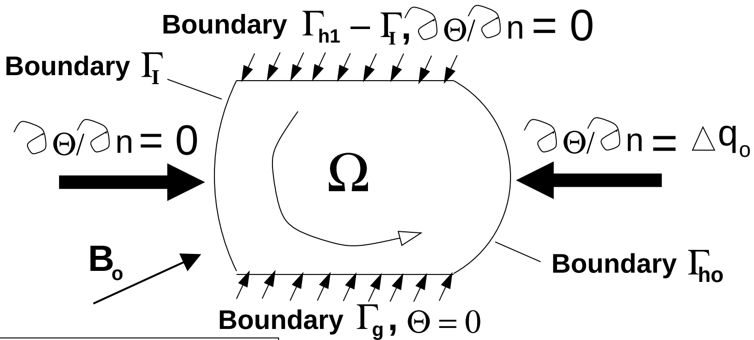
Determined through the solution of an appropriate linear continuum sensitivity problem

Expression for the exact gradient is determined through the solution of an appropriate continuum adjoint problem and is given by

$$\boxed{J'(q) = \psi(x,t;q_o) \quad \text{for } (x,t) \in (\Gamma_{h0} \times [0,\, t_{max}])}$$

Adjoint temperature field

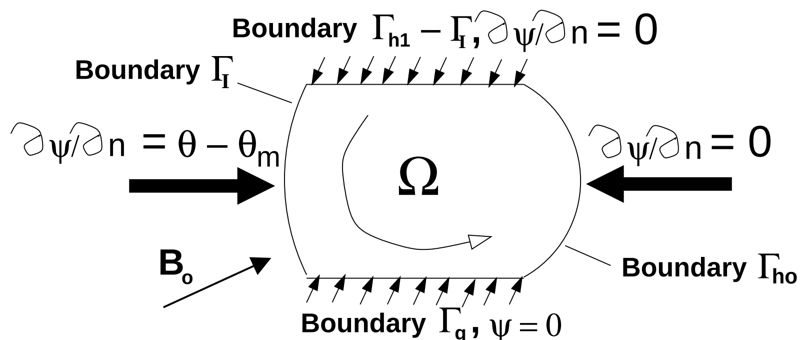
SENSITIVITY MAGNETO-CONVECTION PROBLEM



Sensitivity velocity field $\boldsymbol{V}$
Sensitivity temperature field Θ
Sensitivity electric potential field Φ

On all boundaries
$\boldsymbol{V} = 0, \partial\Phi/\partial n = 0$

ADJOINT MAGNETO-CONVECTION PROBLEM



Adjoint velocity field ϕ
Adjoint temperature field ψ
Adjoint electric potential field η

On all boundaries
$\phi = 0, \partial\eta/\partial n = 0$

HIGHLIGHTS OF THE FINITE ELEMENT METHOD

- PSPG/SUPG stabilized fluid flow formulation with additional terms from buoyancy and Lorentz body force terms
- Consistent SUPG formulation for heat equation
- Decoupled solution methodology for the subproblems at a given time step
- One-step time integration (T1 formulation) for the fluid flow problem and Newmark scheme for the heat equation
- Symmetrized coefficient matrix for the pressure equation and use of lumped mass matrix
- Flow fields are obtained after 2 passes per time step
- LU-factorization of the stiffness matrix is performed only once for fixed domain problems
- For deforming domain problems (applications in solidification process design):
  - Preconditioned Bi-CGSTAB algorithm is employed
  - LU-factorization of the stiffness matrix is calculated at regular time intervals and is used as an effective preconditioner in Bi-CGSTAB algorithm
- Identical time integration and finite element solution procedures is used for direct, adjoint, and sensitivity problems
- "Backward in time" solution of the adjoint problem requires that the entire history of the direct problem solution be stored

H^1 REGULARIZATION TO HANDLE RANDOM ERRORS IN INPUT TEMPERATURE DATA

MODIFIED COST FUNCTIONAL

$$J(q_o) = \frac{1}{2}\left\| \theta(x,t;q_o) - \theta_m(x,t) \right\|^2_{L_2(\Gamma_I \times [0,\, t_{max}])} + \frac{\gamma}{2}\left\| q_o \right\|^2_{L_2(\Gamma_{ho} \times [0,\, t_{max}])} + \frac{\gamma}{2}\left\| \nabla q_o \right\|^2_{L_2(\Gamma_{ho} \times [0,\, t_{max}])}$$

Regularization parameter

DIRECTIONAL DERIVATIVE

$$D_{\Delta q_o} J(q_o) \equiv (\Psi(x,t;q_o), \Delta q_o)_{L_2(\Gamma_{ho} \times [0,\, t_{max}])} + \gamma(q_o(x,t), \Delta q_o)_{L_2(\Gamma_{ho} \times [0,\, t_{max}])} + \gamma(\nabla q_o(x,t), \nabla(\Delta q_o))_{L_2(\Gamma_{ho} \times [0,\, t_{max}])}$$

EXPRESSION FOR MODIFIED GRADIENT

$$J'(q_o) = z + \gamma q_o$$

Solution of the variational equation

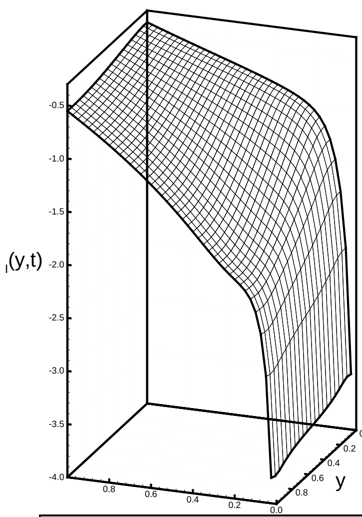
$$-\Delta z(x,t) + z(x,t) = \Psi(x,t;q_o)$$

NUMERICAL EXAMPLE

- **Two-dimensional unit square cavity**

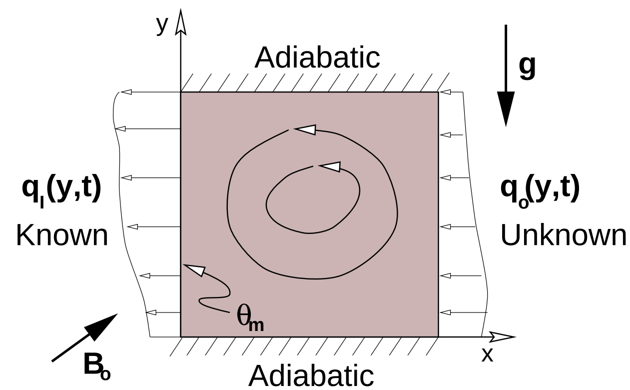
DIMENSIONLESS PARAMETERS

- **Prandtl number (Pr) 0.01**
- **Rayleigh number (Ra) 2×10^{4}**
- **Hartmann number (Ha) 75 (applied at 45° inclination to the x-axis)**
- **Initial temperature (θ_i) 1.0**
- **Measured temperature (θ_m) 1.0**



Given heat flux history q_I applied on the left vertical wall of the cavity

Calculated from the solution of a direct magneto-convection problem with flux $q_o(y,t) = 1 - t$ applied on the right wall and the left wall maintained at θ_m

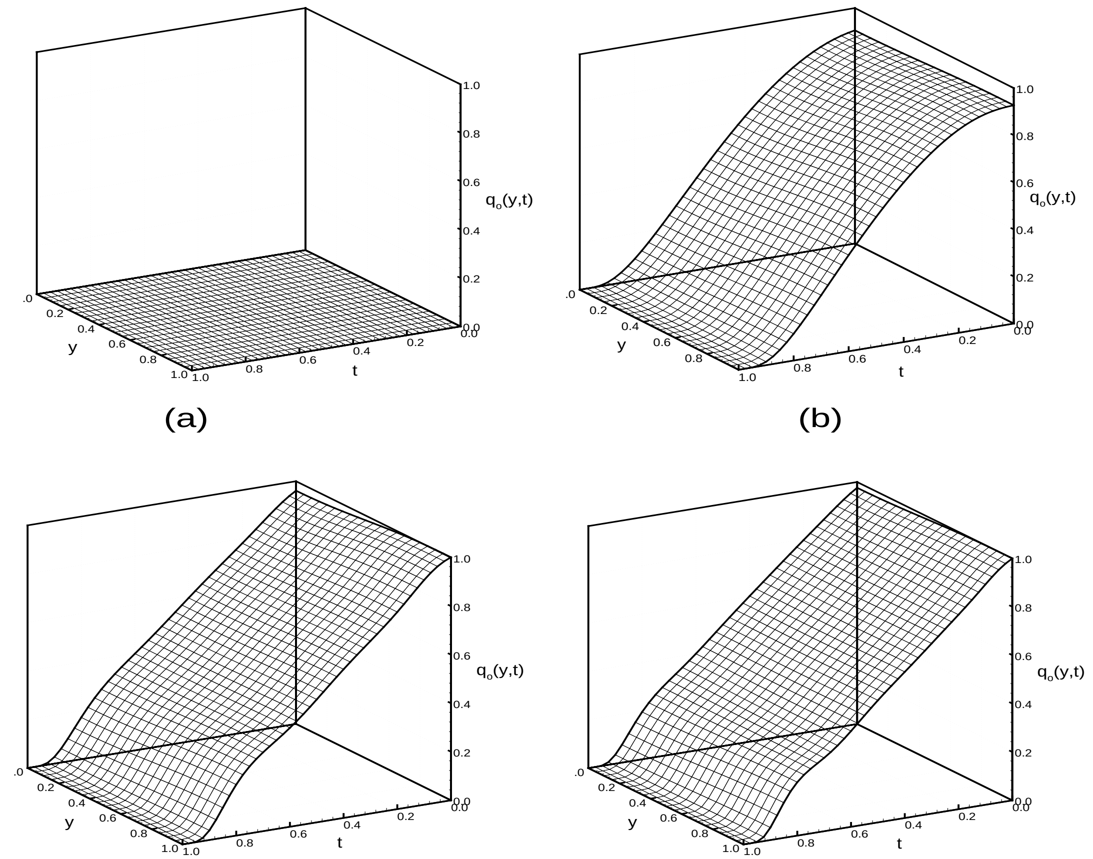


OBJECTIVE OF THE INVERSE PROBLEM

RECONSTRUCT THE EXACT FLUX SOLUTION $\overline{q}_o(y,t) = 1 - t$ USING THE OVERSPECIFIED CONDITIONS $q_I(y,t) = 1 - t$ AND θ_m, STARTING FROM ANY ARBITRARY INITIAL GUESS HEAT FLUX

CONVERGENCE OF THE HEAT FLUX SOLUTION

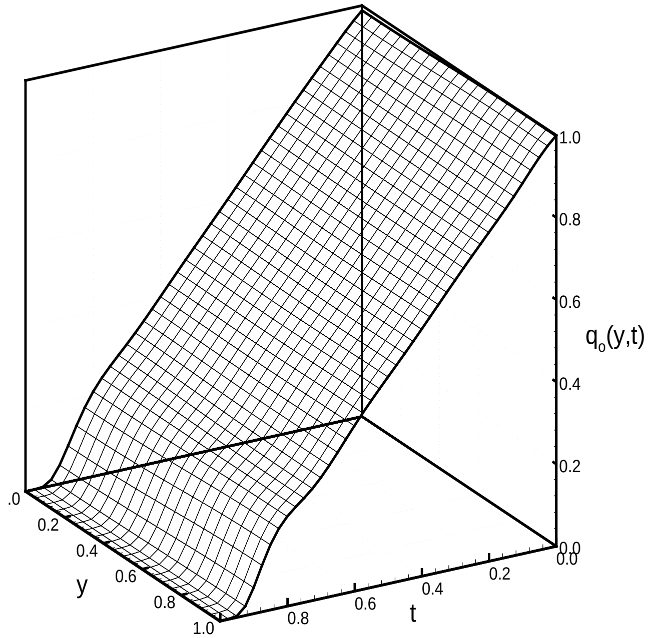
INITIAL GUESS $q_o^0 = 0$ AND INTERMEDIATE FLUXES $q_o^k(y,t)$ AT ITERATIONS 1, 10, 20



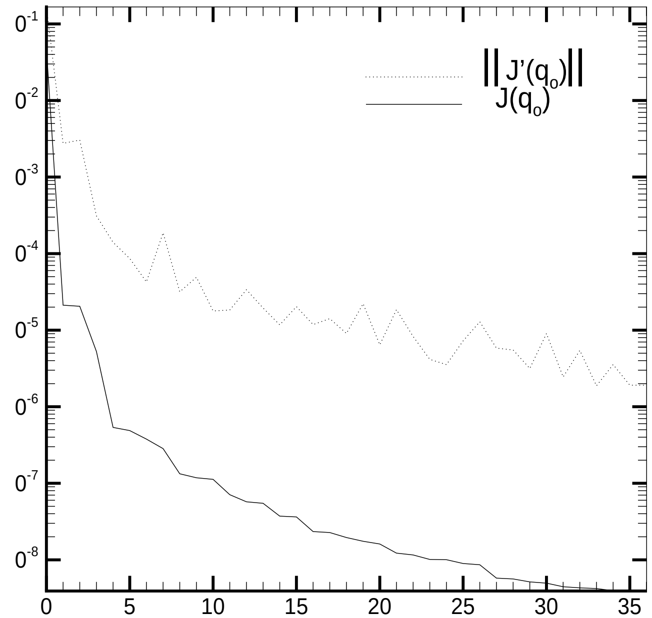
(a) (b)

(c) (d)

CONVERGENCE OF THE CONJUGATE GRADIENT METHOD



OPTIMAL HEAT FLUX DISTRIBUTION CALCULATED AT THE 35th CONJUGATE-GRADIENT ITERATION



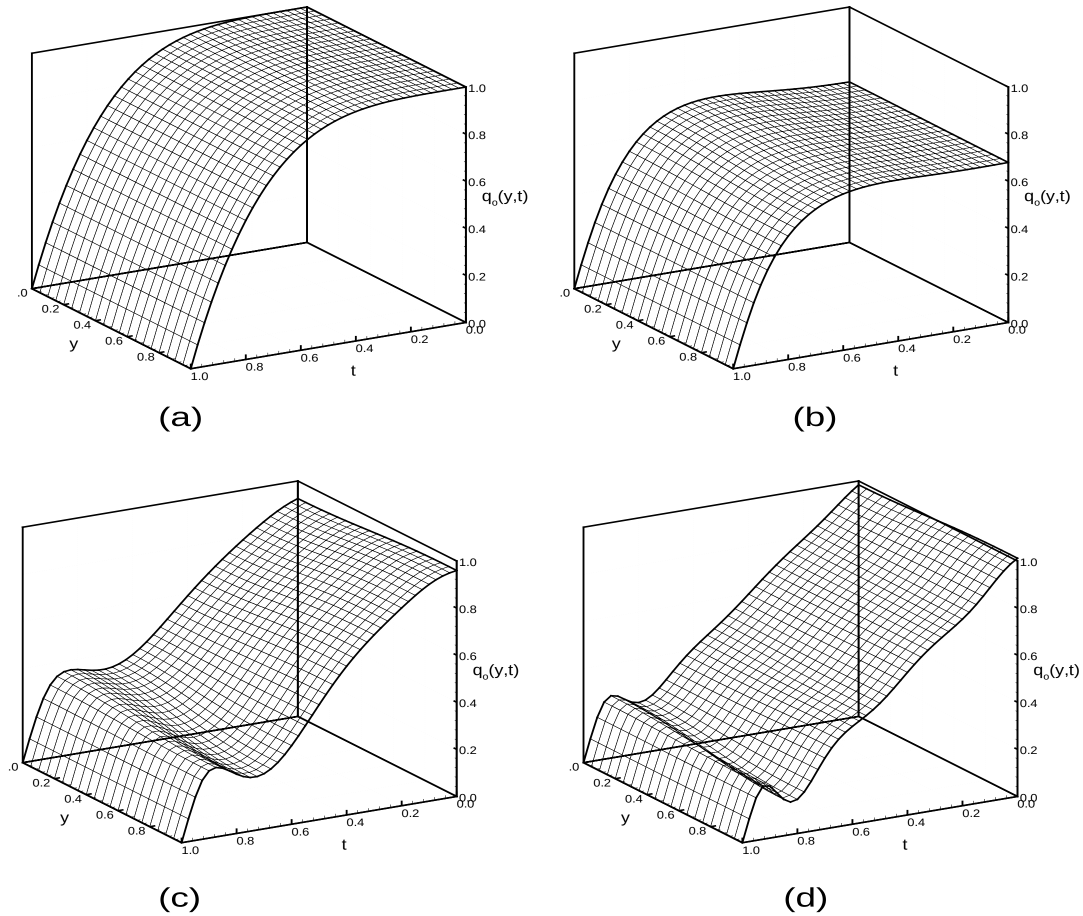
VARIATION OF THE COST FUNCTIONAL AND THE NORM OF THE GRADIENT WITH CG ITERATIONS

IMPORTANT TRENDS IN THE NUMERICAL SOLUTION

- **Monotonic reduction of the cost functional**
- **Norm of the gradient decreases with CG iterations (but not monotonically)**
- **Maximum inaccuracy in the calculated solution is at $t = t_{max}$**

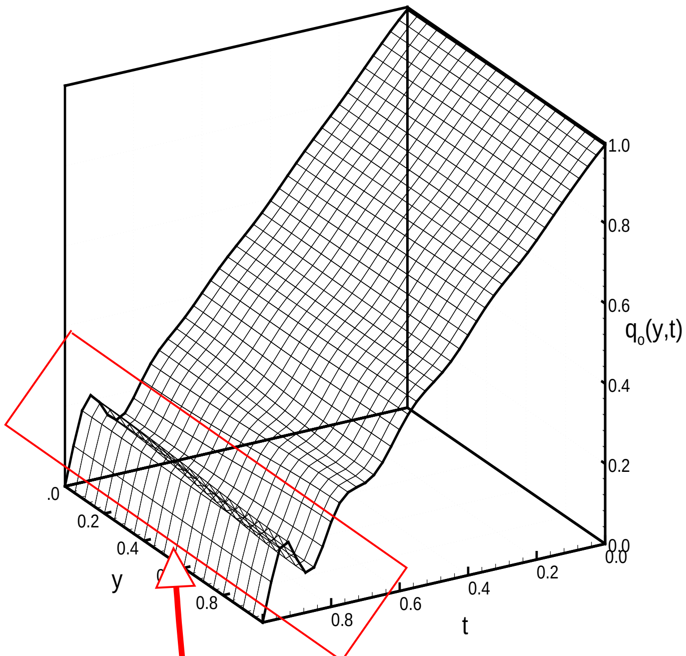
EFFECTS OF THE INITIAL GUESS HEAT FLUX

INITIAL GUESS $q_o^0 = 1 - t^4$ AND INTERMEDIATE FLUXES $q_o^k(y,t)$ AT ITERATIONS 1, 5, 20



CONVERGENCE OF THE CG METHOD FOR INITIAL GUESS

$q_o^0(y,t) = 1 - t^4$



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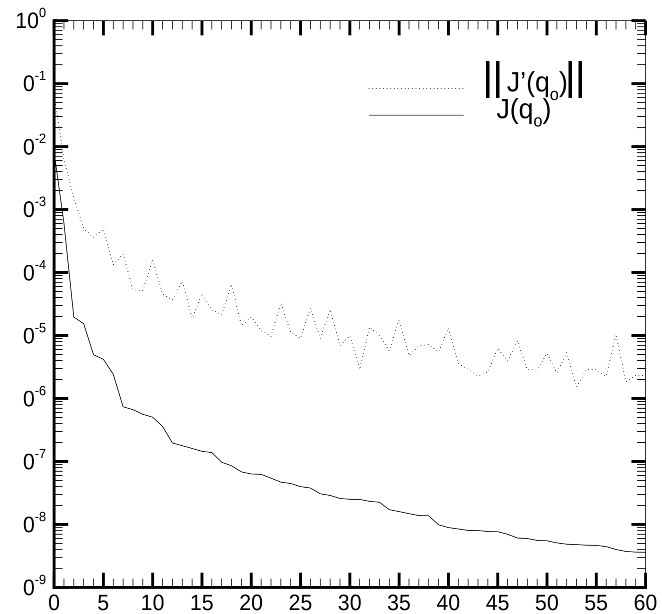
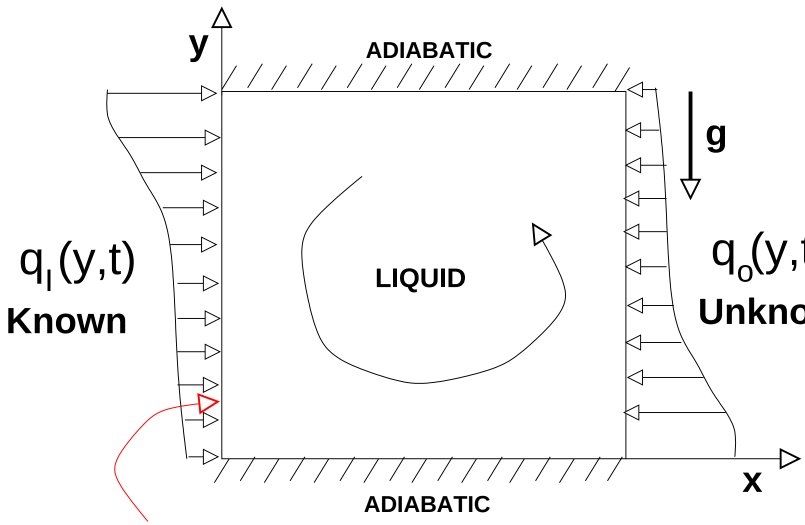


VARIATION OF THE COST FUNCTIONAL AND THE NORM OF THE GRADIENT WITH CG ITERATIONS

NOTICE THE PROMINENT DEVIATION OF THE CALCULATED SOLUTION FROM THE EXACT SOLUTION $\overline{q}_o = 1 - t$

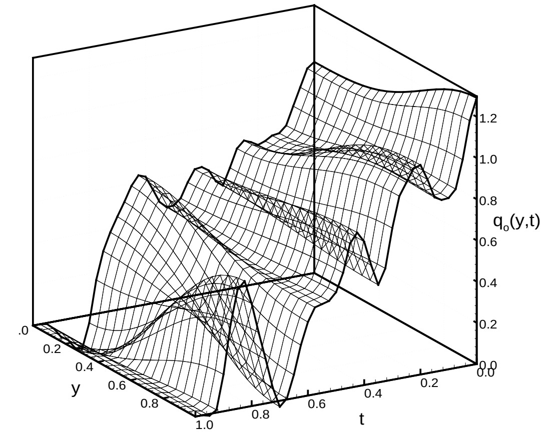
END-CONDITION PROBLEM ASSOCIATED WITH THE ADJOINT METHOD

EFFECTS OF LARGE MEASUREMENT ERRORS

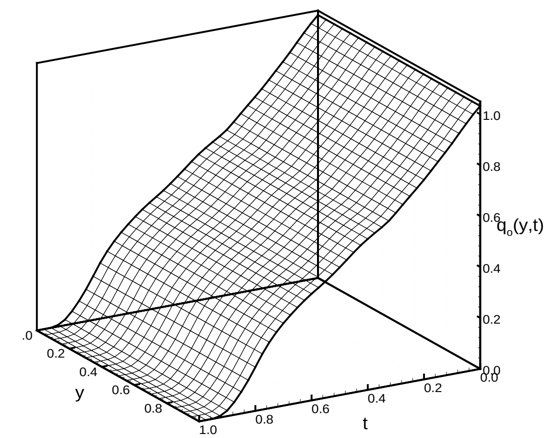


$$\theta_m(y,t) = \theta_m(y,t) + \varepsilon\omega_n$$

0.0 0.05 Uniformly distributed random numbers in the interval [-1, 1]



CALCULATED HEAT FLUX SOLUTION AT THE 30th ITERATION (NO REGULARIZATION)



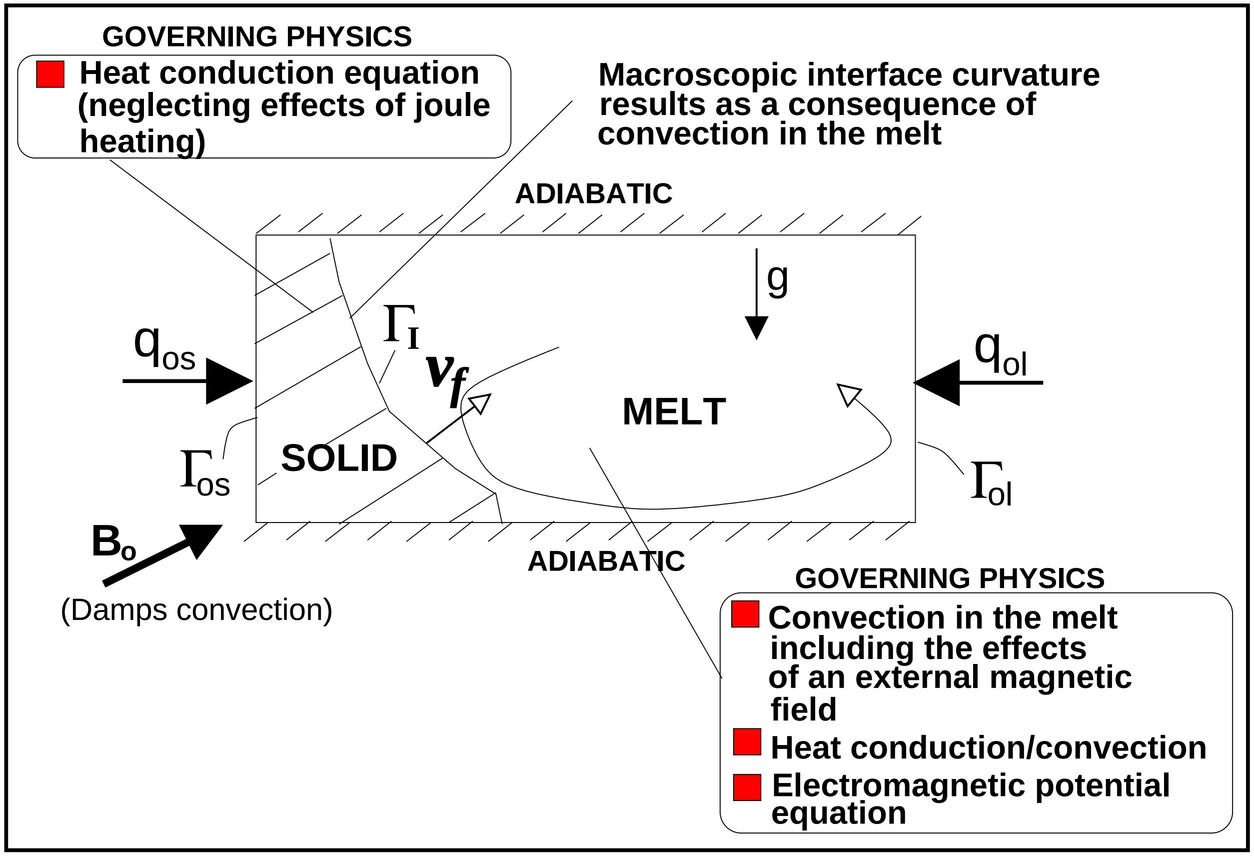
CALCULATED HEAT FLUX SOLUTION AT THE 30th ITERATION (WITH REGULARIZATION)

OBJECTIVE OF THE INVERSE PROBLEM

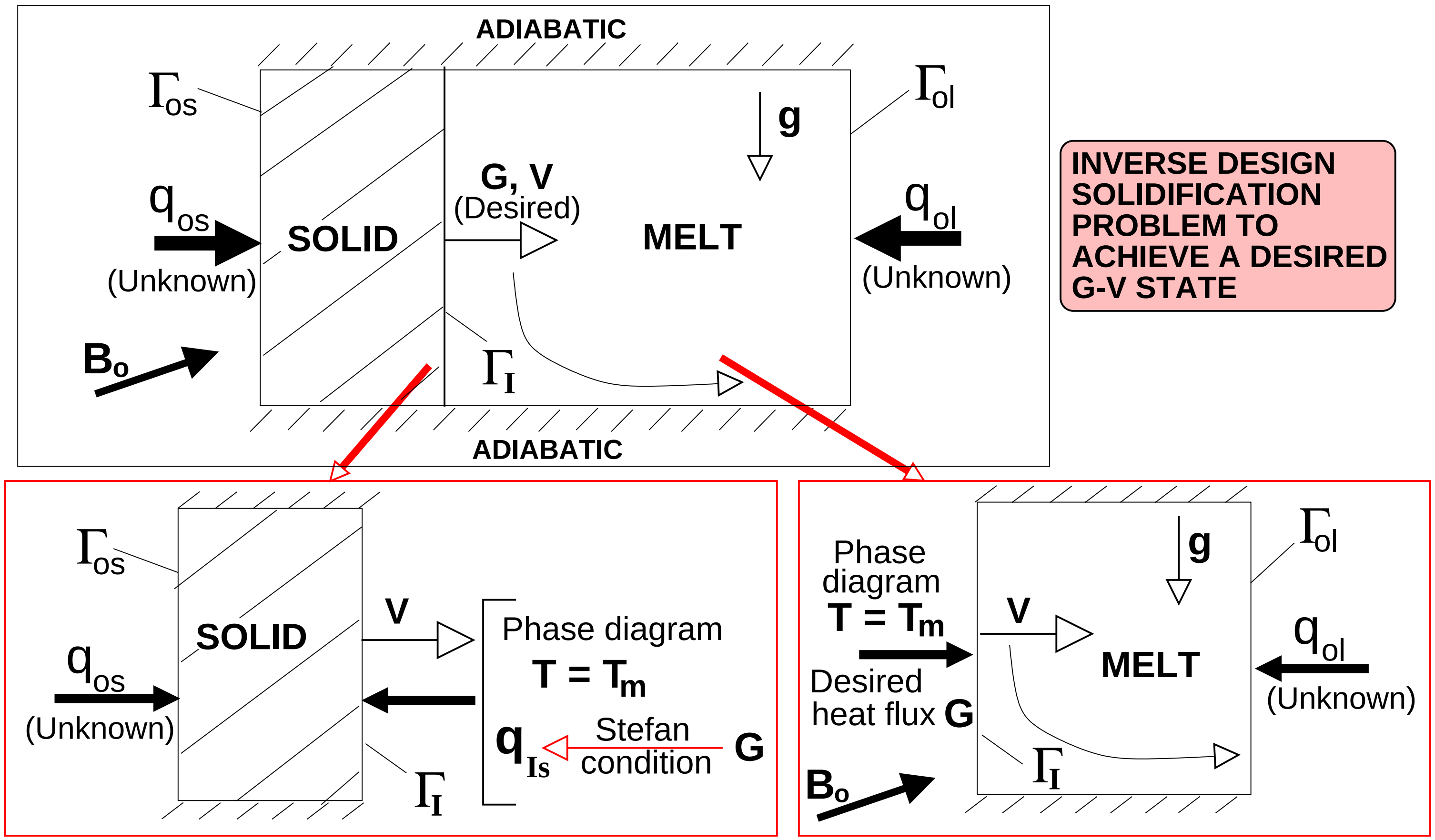
Reconstruct a close approximation of the optimal solution $\overline{q}_o = 1 - t$ using the inexact temperature measurements and the given flux $q_I(y,t)$, starting from any arbitrary initial guess heat flux

APPLICATIONS TO DESIGN OF DIRECTIONAL SOLIDIFICATION PROCESSES

PHYSICAL MODEL



APPLICATIONS TO DESIGN OF DIRECTIONAL SOLIDIFICATION PROCESSES



INVERSE HEAT CONDUCTION PROBLEM

INVERSE MAGNETO-CONVECTION PROBLEM

SUMMARY

- Systematic continuum adjoint formulation for the inverse design of convective systems under the influence of an external magnetic field
- Accurate reconstruction of heat fluxes for several example problems (including 3D systems) with exact measurement data
- Demonstration of the need for regularization for problems with large measurement errors
- H^1 regularized inverse design formulation for problems with large input errors
- Application of the developed methods to thermal design of directional solidification processes such that a desired G-V state is achieved