A spectral stochastic approach to the inverse heat conduction problem

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Abstract

A spectral stochastic approach to the inverse heat conduction problem (IHCP) is presented. In IHCP, one computes an unknown boundary heat flux from given temperature history data at a sensor location. In the stochastic inverse heat conduction problem (SIHCP), the full statistics of the boundary heat flux are computed given the stochastic nature of the temperature sensor data and in general accounting for uncertainty in the material data and process conditions. The governing continuum equations are solved using the spectral stochastic finite element method (SSFEM). The stochasticity of inputs is represented spectrally by employing orthogonal polynomials as the trial basis in the random space. Solution to the ill-posed SIHCP is then sought in an optimization sense in a function space that includes the random space. The gradient of the objective function is computed in a continuum sense using an adjoint framework. Finally, an example is presented in the solution of a

Preprint submitted to Elsevier Science *25 March 2003*

one-dimensional stochastic inverse heat conduction problem in order to highlight the methodology and potential applications of the proposed techniques.

Key words: Stochastic inverse heat conduction (SIHCP), optimization, adjoint methods, spectral stochastic finite element method (SSFEM), uncertainty, robust design.

1 Stochastic inverse heat conduction problem

1.1 Introduction

Mathematical models used to describe the behavior of physical systems should correspond well with experimentally observed facts. Since experimental results are always polluted with uncertainties, a good model for a physical system should be inherently probabilistic and governed by stochastic PDE's. In the spectral stochastic finite element method (SSFEM), the solution process is expressed in the form of a Fourier like expansion in random variables [1], [2]. A stochastic process $w(\mathbf{x}, t, \theta)$ is expressed in terms of a denumerable set of orthogonal random variables in the form of a series expansion:

$$w(\mathbf{x}, t, \theta) = \sum_{i=0}^{\infty} \mu_i(\theta) g_i(\mathbf{x}, t) \tag{1.1}$$

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where θ is a member of the sample space of elementary events $\mathcal{S}$, $\{\mu_i(\theta)\}_{i=0}^{\infty}$ is a set of orthogonal random variables and $\{g_i(\mathbf{x},t)\}_{i=0}^{\infty}$ is a set of deterministic functions.

This type of series expansion of the random process can be viewed as an abstract discretization of the process in the random dimension. Also, $\{\mu_i(\theta)\}_{i=0}^{\infty}$ spans $\mathcal{S}$. Thus such a series expansion can be considered as the direct sum of orthogonal projections of $w(\mathbf{x},t,\theta)$ onto the basis of the space $\mathcal{S}$.

In a SSFEM implementation of a continuum system, the system parameters are modelled as random processes using a Karhunen-Loève or polynomial chaos expansion usually. The stochastic finite element analysis is then introduced in order to compute the output response quantification.

In this work, interest is given in using the SSFEM to address a stochastic version of the inverse heat conduction problem. A typical inverse problem comprises of governing continuum equations with insufficient or no boundary conditions in part of the boundary and overspecified or extra boundary conditions on another part of the boundary or within the domain Zabaras et al. [3]. These ill-posed problems can be restated as functional optimization problems where a quasi-solution is sought. The stochastic inverse heat conduction problem (SIHCP) is solved by extending the functional theories for deterministic inverse heat conduction analysis to the stochastic case. The developed techniques can be applied to flux computation problems given the statistics of sensor temperature data or given the statistics of a desired temperature response at some point(s) in the domain.

1.2 Problem definition

Let Ω be a region in $\mathbb{R}^d$ bounded by Γ with partitions Γ_o and Γ_h ; $\Gamma = \Gamma_o \cup \Gamma_h$, $\Gamma_o \cap \Gamma_h = \{\emptyset\}$. The thermal conductivity k and heat capacity C of the medium are random. It is further assumed that the heat flux on Γ_h is known. The heat flux on the boundary Γ_o is unknown and is to be constructed given the random sensor temperature readings $Y(\mathbf{x}, t, \theta)$ with all statistics on a boundary Γ_I (in the interior of Ω). In robust design context, one can describe $Y(\mathbf{x}, t, \theta)$ as the desired temperature statistics on Γ_I. The system of equations summarizing this SIHCP are given below:

$$C\frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T), \quad (\mathbf{x}, t, \theta) \in \Omega \times [0, t_{max}] \times \mathcal{S} \tag{1.2}$$

$$T(\mathbf{x}, 0, \theta) = T_o(\mathbf{x}, \theta), \qquad (\mathbf{x}, \theta) \in \Omega \times \mathcal{S} \tag{1.3}$$

$$k\frac{\partial T}{\partial n} = f(\mathbf{x}, t, \theta), \quad (\mathbf{x}, t, \theta) \in \Gamma_h \times [0, t_{max}] \times \mathcal{S} \tag{1.4}$$

$$k\frac{\partial T}{\partial n} = q_o(\mathbf{x}, t, \theta), \quad (\mathbf{x}, t, \theta) \in \Gamma_o \times [0, t_{max}] \times \mathcal{S} \quad (q_o \text{ unknown}) \tag{1.5}$$

$$T(\mathbf{x}, t, \theta; q_o(\mathbf{x}, t, \theta)) \simeq Y(\mathbf{x}, t, \theta), \quad (\mathbf{x}, t, \theta) \in \Gamma_I \times [0, t_{max}] \times \mathcal{S} \tag{1.6}$$

In this work, it is assumed that a quasi-solution to the inverse problem exists in the sense of Tichonov. In particular, we look for a flux $\bar{q}_o(\mathbf{x}, t, \theta) \in L_2(\Gamma_o \times$

$[0, t_{max}] \times \mathcal{S})$ such that

$$\mathcal{J}(\bar{q}_o) \leq \mathcal{J}(q_o), \quad \forall\, q_o \in L_2(\Gamma_o \times [0, t_{max}] \times \mathcal{S}) \tag{1.7}$$

where the objective function $\mathcal{J}(q_0)$ is defined as,

$$\mathcal{J}(q_0) = \frac{1}{2} \|T(\mathbf{x}, t, \theta; q_0) - Y(\mathbf{x}, t, \theta)\|^2_{L_2(\Gamma_I \times [0, t_{max}] \times \mathcal{S})} \tag{1.8}$$

$$= \frac{1}{2} \int_{\Gamma_I} \int_0^{t_{max}} \int_{\mathcal{S}} [T(\mathbf{x}, t, \theta; q_0) - Y(\mathbf{x}, t, \theta)]^2 d\Gamma \mathrm{d}t \mathrm{d}P \tag{1.9}$$

$T(\mathbf{x}, t, \theta; q_o) \equiv T(\mathbf{x}, t, \theta; q_o(\mathbf{x}, t, \theta))$ is the solution of the parametric direct problem and $\mathrm{d}P$ is a probability measure in $\mathcal{S}$ (Zabaras et al. [4]). The gradient of the objective function $\mathcal{J}'(q_o)$ is calculated by using the definition of the directional derivative of $\mathcal{J}(q_o)$

$$\begin{aligned} D_{\Delta q}\mathcal{J}(q_o) &\equiv (\mathcal{J}'(q_o), \Delta q_o)_{L_2(\Gamma_o \times [0, t_{max}] \times \mathcal{S})} \\ &= ([T(\mathbf{x}, t, \theta; q_0) - Y(\mathbf{x}, t, \theta)], \Theta(\mathbf{x}, t, \theta; q_o, \Delta q_o))_{L_2(\Gamma_I \times [0, t_{max}] \times \mathcal{S})} \end{aligned} \tag{1.10}$$

The definitions of the sensitivity field $\Theta(\mathbf{x}, t, \theta; q_o, \Delta q_o)$ and the corresponding adjoint variable $\phi(\mathbf{x}, t, \theta; q_0)$ are summarized in Boxes I and II respectively, with details of their derivation given in Zabaras et al. [4]. It can be shown that the gradient of the objective function in $L_2(\Gamma_o \times [0, t_{max}] \times \mathcal{S})$ is given as follows:

$$\mathcal{J}'(q_o) = \phi(\mathbf{x}, t, \theta; q_o) \quad (\mathbf{x}, t, \theta) \in (\Gamma_o \times [0, t_{max}] \times \mathcal{S}) \tag{1.11}$$

The SIHCP is implemented as an optimization problem in functional spaces. The non-linear conjugate gradient method is used for the optimization process

Zabaras et al. [4].

Box I: Sensitivity problem to define $\Theta(\mathbf{x}, t, \theta; q_0, \Delta q_0)$

$$C\frac{\partial \Theta}{\partial t} = \nabla \cdot (k\nabla\Theta), \quad (\mathbf{x}, t, \theta) \in (\Omega \times [0, t_{max}] \times \mathcal{S}) \tag{1.12}$$

$$\Theta(\mathbf{x}, 0, \theta; q_o, \Delta q_o) = 0, \quad (\mathbf{x}, \theta) \in (\Omega \times \mathcal{S}) \tag{1.13}$$

$$k\frac{\partial \Theta}{\partial n}(\mathbf{x}, t, \theta; q_o, \Delta q_o) = \Delta q_o(\mathbf{x}, t, \theta), \quad (\mathbf{x}, t, \theta) \in (\Gamma_o \times [0, t_{max}] \times \mathcal{S}) \tag{1.14}$$

$$k\frac{\partial \Theta}{\partial n}(\mathbf{x}, t, \theta; q_o, \Delta q_o), = 0, \quad (\mathbf{x}, t, \theta) \in (\Gamma_h \times [0, t_{max}] \times \mathcal{S}) \tag{1.15}$$

Box II: Adjoint problem to define $\phi(\mathbf{x}, t, \theta; q_0(\mathbf{x}, t, \theta))$

$$C\frac{\partial \phi}{\partial t} = -\nabla \cdot (k\nabla\phi) \quad (\mathbf{x}, t, \theta) \in (\Omega \times [0, t_{max}] \times \mathcal{S}) \tag{1.16}$$

$$\phi(\mathbf{x}, t_{max}, \theta) = 0, \quad (\mathbf{x}, \theta) \in (\Omega \times \mathcal{S}) \tag{1.17}$$

$$k\frac{\partial \phi}{\partial n}(\mathbf{x}, t, \theta) = 0, \quad (\mathbf{x}, t, \theta) \in (\Gamma \times [0, t_{max}] \times \mathcal{S}) \tag{1.18}$$

$$\left[k\frac{\partial \phi}{\partial n}(\mathbf{x}, t, \theta)\right]_{\Gamma_I} = T(\mathbf{x}, t, \theta; q_0) - Y(\mathbf{x}, t, \theta), \; (\mathbf{x}, t, \theta) \in (\Gamma_I \times [0, t_{max}] \times \mathcal{S}) \tag{1.19}$$

2 Some implementation issues

Following our earlier work in Zabaras et al. [5], an object-oriented implementation of the stochastic inverse heat conduction problem was introduced. Such an implementation takes advantage of the similar structure of the direct, sensitivity and adjoint stochastic problems.

As discussed in Ghanem et al. [6], the SSFEM solution of PDE's requires solving a block matrix system of equations. The solution technique is based on variations of the block-Jacobi algorithm that exploits the block symmetry and sparsity structure of the block matrix. Detailed accounting on application of various forms of stochastic boundary conditions within the SSFEM framework are discussed in Badri Narayanan and Zabaras [4].

The general form of the SSFEM system of equations requires the computation of statistical averages of the Wiener-chaos polynomials ψ_k e.g. of $\langle \xi_i \psi_j \psi_k \rangle$ (see Reference [2]). These averages have been computed separately and stored in a library for future usage. It is to be noted that each random field is expressed in its spectral series representation. This requires the storage of all the series coefficients at each node in the computational domain.

3 Numerical Examples

A one dimensional SIHCP is solved. The computational domain considered is a $[0, 1]$ bar, comprising of 40 linear elements. The dimensionless material data and process conditions are as follows.

$$\hat{k} = 1 + 0\xi_1(\theta), \qquad \hat{c} = 1 + 0\xi_2(\theta) \tag{3.1}$$

$$Y(0.5, \hat{t}, \theta) = e^{(-\frac{\pi^2}{4}\hat{t})} \sin(\pi/4)\,(1 + 0.1\psi_1) \tag{3.2}$$

$$\hat{T}_i = \hat{T}_o(\hat{x}, 0, \theta) = \sin(\pi\hat{x}/2)[1 + 0\psi_1(\theta)] \tag{3.3}$$

$$\hat{k}\frac{\partial \hat{T}}{\partial \hat{x}}(1, \hat{t}, \theta) = 0 \tag{3.4}$$

$$\hat{k}\frac{\partial \hat{T}}{\partial \hat{x}}(0, \hat{t}, \theta) = q(\hat{t}, \theta) \quad \text{unknown heat flux} \tag{3.5}$$

where in general $\hat{k} = k/k_{\text{mean}}$, $\hat{C} = C/C_{\text{mean}}$ and $\hat{T}$, $\hat{x}$ and $\hat{t}$ are the non-dimensional temperature, location coordinate and time respectively. These quantities are defined as follows $\hat{T} = (T - T_i)/\Delta T_{\text{ref}}$, $\hat{x} = \frac{x}{L}$ and $\hat{t} = t/(L^2/\alpha)$ (α=diffusivity). $\xi_1(\theta)$ and $\xi_2(\theta)$ are uncorrelated $N(0,1)$ random variables. Note from Equation (3.2) that the sensor reading is here assumed to be Gaussian with a coefficient of variation 0.1.

The total time for computation was taken to be $\hat{t} \in [0,2]$, the time-step $\Delta\hat{t} = 0.01$ and the initial guess heat flux as $q_o^0 = 0$. The mean optimal heat flux $\bar{q}_o$ has a closed form

$$\bar{q}_o(\hat{t}) = -\frac{\pi}{2} e^{(-\frac{\pi^2}{4}\hat{t})} \tag{3.6}$$

The results are shown in Figures 1-3. It can be noted here that since the material properties $\hat{k}$ and $\hat{C}$ are assumed to be deterministic, the only uncertainty being in the sensor readings $Y(x,t,\theta)$, we expect the flux to reproduce similar uncertainty (i.e. to be Gaussian). This indeed was the case.

Since the error in the heat flux is accounted for by the standard deviation *as opposed to the deterministic case where we assume the error in heat flux to be bounded below by the error in sensor reading*, the objective function smoothly decays with iteration index. The gradient of the objective function is also observed to be smooth and monotonic. The statistics of the conjugate gradient method are shown in Figure 4. The simulation took about 12 minutes on a Pentium 1 Ghz workstation.

4 Discussion

The conjugate gradient approach to inverse heat conduction problems was extended here to include the random dimension. The one dimensional SIHCP was solved and the unknown boundary heat flux was reconstructed with all the statistics. The code developed for this purpose is dimension-independent and two dimensional stochastic inverse heat conduction problems were also tested. This approach provides a robust design, since we can describe the confidence within which the applied heat flux should be in order to achieve the desired temperature statistics at sensor points. This is of great importance since the method is orders of magnitude faster (about 100 times) than the conventional Monte Carlo techniques currently applied for robust design purposes. This method also extracts the higher moments of flux, this is done by enlarging the space where we look for the solution. This is opposed to the deterministic case, where L_2 optimization techniques do not estimate moments other than mean and standard deviation. The novel functional approach used to solve the SIHCP can easily be extended to the design of complex systems that involve fluid flow and other transport processes. Research is carried on in that regard.

5 ACKNOWLEDGEMENTS

The work presented here was funded in part by the Computational Mathematics program of the Air Force Office of Scientific Research (grant F49620-

00-1-0373) and by the NASA Office of Biological and Physical Research (grant 98-HEDS-05) . This research was conducted using the resources of the Cornell Theory Center, which receives funding from Cornell University, New York State, federal agencies, and corporate partners.

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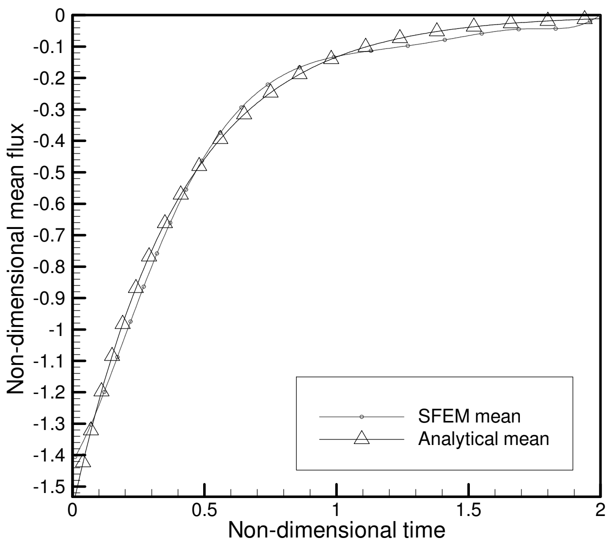

Fig. 1. Computed optimal heat flux mean compared with the analytical mean.

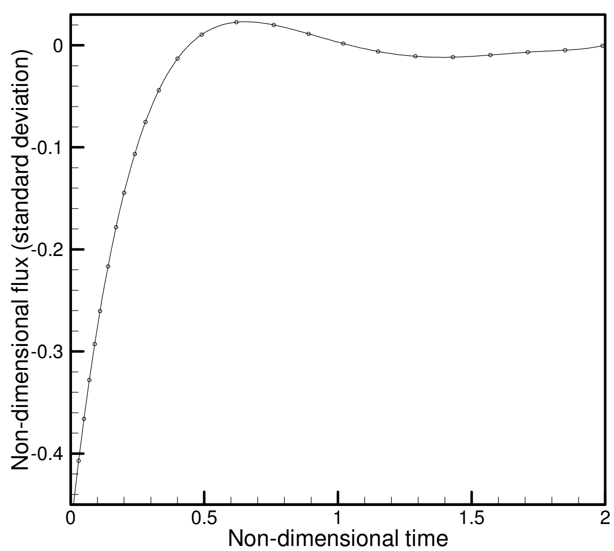

Fig. 2. Computed optimal heat flux standard deviation.

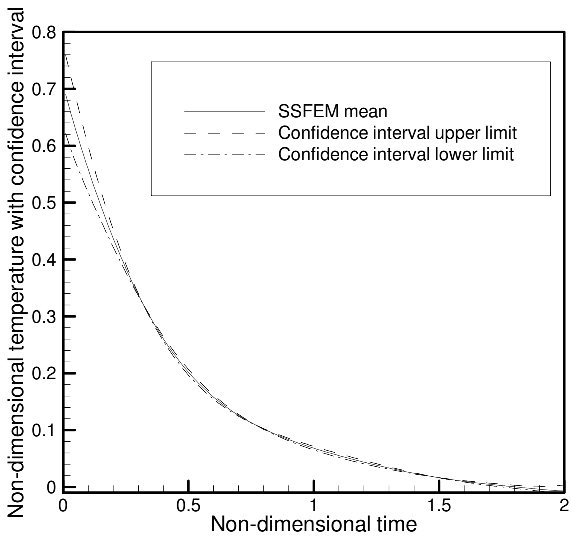

Fig. 3. Computed temperature at the sensor location using the optimal heat flux. The mean values together with the computed confidence interval are shown.

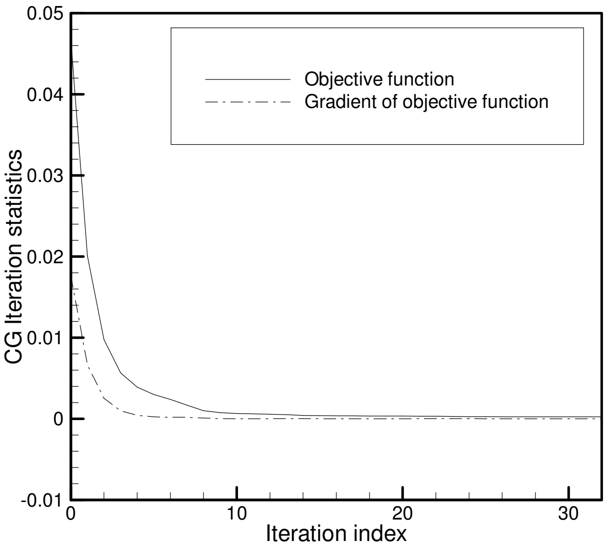

Fig. 4. The objective function and gradient of the objective function versus CGM iteration number.