Stochastic Galerkin/Support space method

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SUPPORT-SPACE [STOCHASTIC GALERKIN]

- Let stochastic inputs be represented by ON random variables $(\xi_1, \ldots, \xi_N)$ with a joint PDF $f_\xi(\xi_1, \ldots, \xi_N)$
- Support space is the region in the span of stochastic input that has a positive PDF

$$A = \left\{ \boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) : f_\xi(\boldsymbol{\xi}) > 0 \right\}$$

- Now consider a discretization of the support space into disjoint finite element subdomains $A = \bigcup_{e=1}^{Nel} A^{(e)}$ with a mesh size h (defined as the maximum diameter of $A^{(e)}$).
- The support-space representation of $X(\boldsymbol{\xi})$ (denoted as $\hat{X}(\boldsymbol{\xi})$) is constructed as a piecewise polynomial of degree q in each element $A^{(e)}$
- Thus the stochastic process can be represented using the basis functions as

$$\hat{X}(\boldsymbol{\xi}) = \sum_{i=1}^{nodes} X_i \phi_i$$

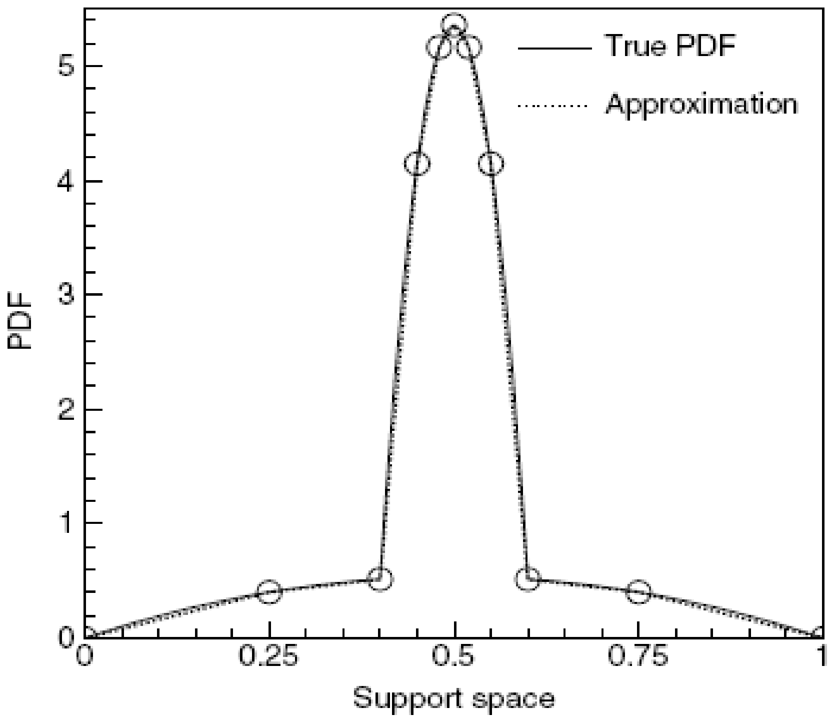
where ϕ_i are the locally supported basis functions and X_i are nodal values.

SUPPORT-SPACE [STOCHASTIC GALERKIN]

- Note that the error in approximation of $X(\xi)$ behaves as follows:

$$\left\| X - \hat{X} \right\|_{L_2} = \left[\int_A (X(\xi) - \hat{X}(\xi))^2 f_\xi(\xi) \mathrm{d}\xi \right] \le h^{q+1}$$

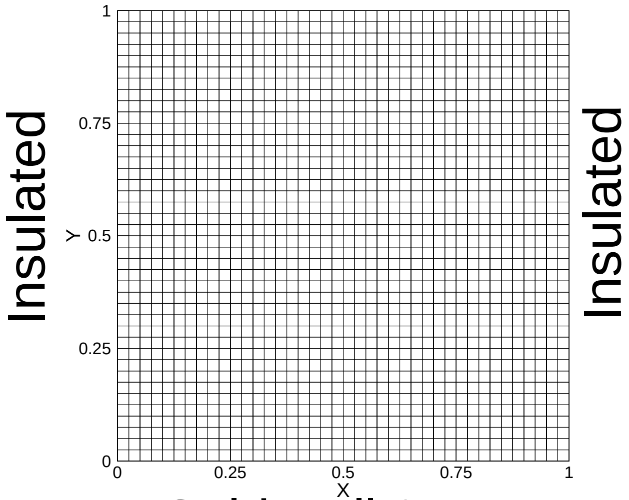
- Example of a 1D input and associated support-space



CAPTURING UNSTABLE EQUILIBRIUM

- 1600 bilinear elements, 5th order Legendre chaos expansion for velocity and pressure, preconditioned GMRES

Cold wall $\theta = 0$



Cold wall $\theta = 1$

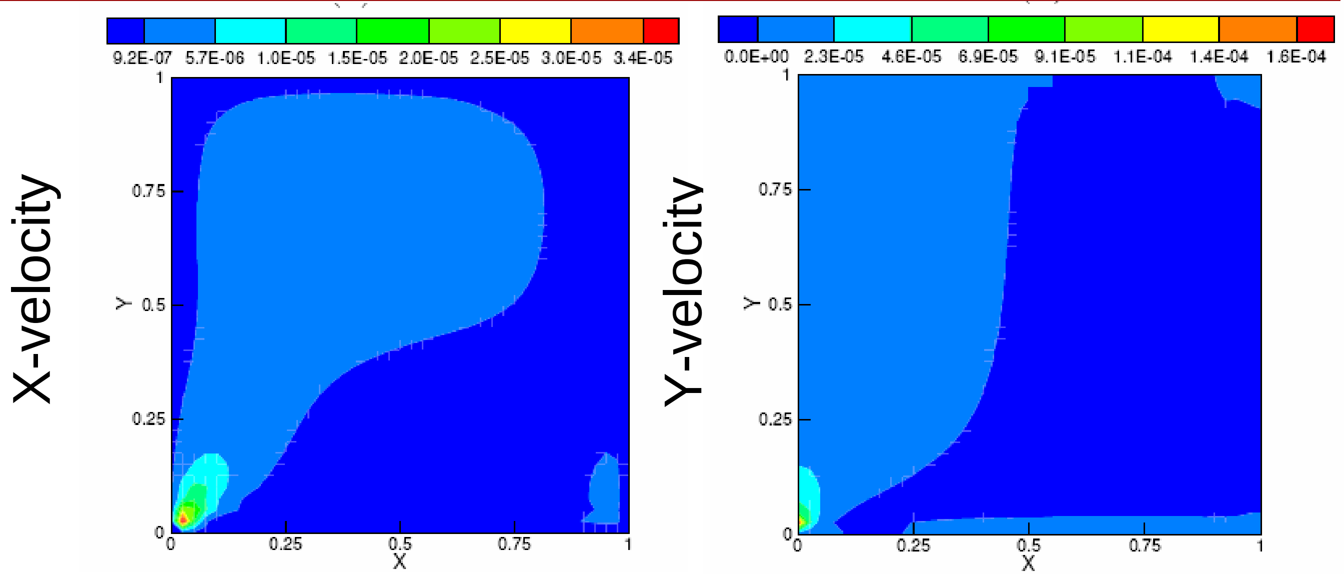
- Time [0,1.5]
- Time step [0.002]
- Rayleigh number unif[1530,1870]
- Prandtl number [6.95]
- Support-space mesh [10 linear elements]

From: Asokan and Zabaras, JCP (2005)

- Simulation around critical Rayleigh number 1700 [Le Maître, JCP (2004)]
- Failure of GPCE approach and use of support-space
- The deterministic velocity contours for Ra= 1530 and Ra= 1870 can be thought of as lower and upper bounds of the flow velocity.

FAILURE OF GPCE

❑ Mean X and Y velocity obtained from GPCE yield unphysical low values



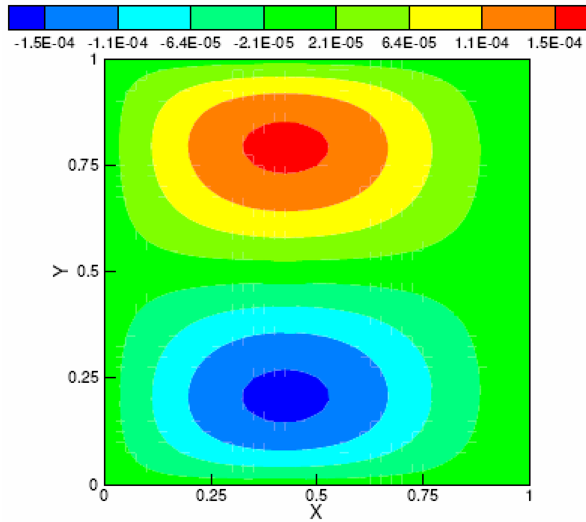
❖ This system is characterized by the critical Rayleigh number of 1700. Below this Rayleigh number, the heat transfer is by conduction and the fluid flow is absent. Above this Rayleigh number, well known Rayleigh-Benard instabilities are initiated. Heat transfer is by conduction and convection.

❖ So the GPCE dose not work in the case of discontinuity.

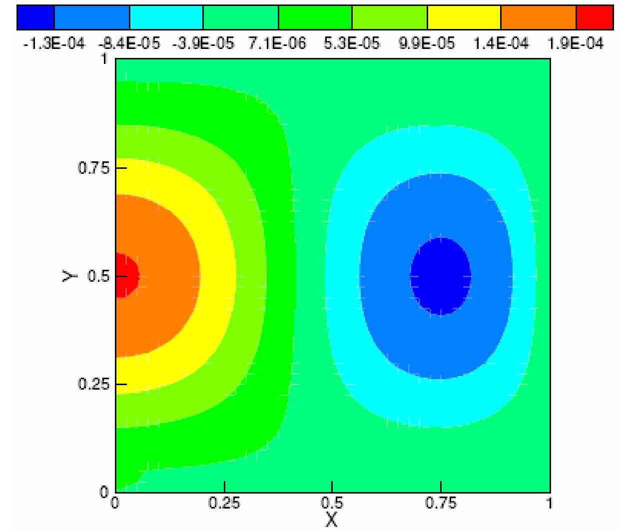
PREDICTION BY SUPPORT-SPACE METHOD

- ❑ X and Y velocity obtained using a support-space method at Ra = 1530

X-velocity

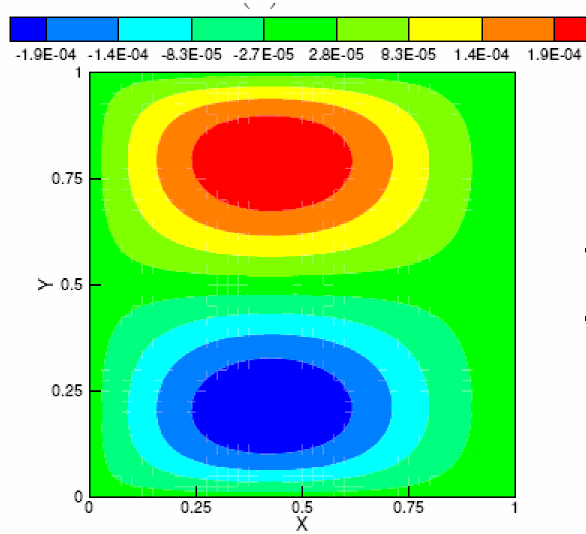


Y-velocity

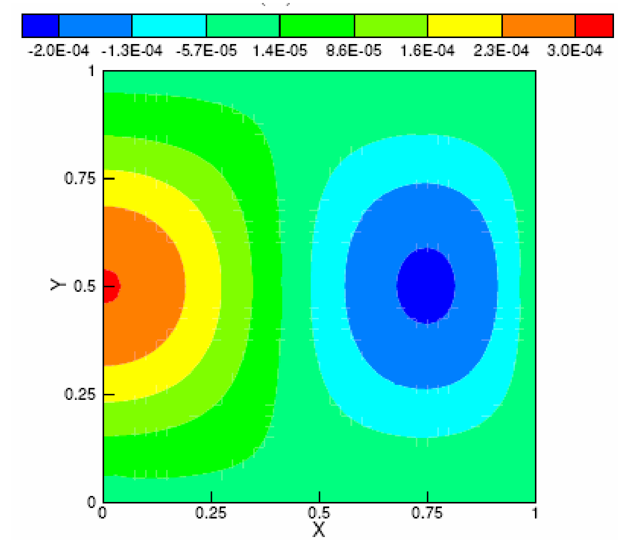


- ❑ Comparative X and Y velocity obtained using a deterministic simulation at Ra = 1530 (the lower limit)

X-velocity

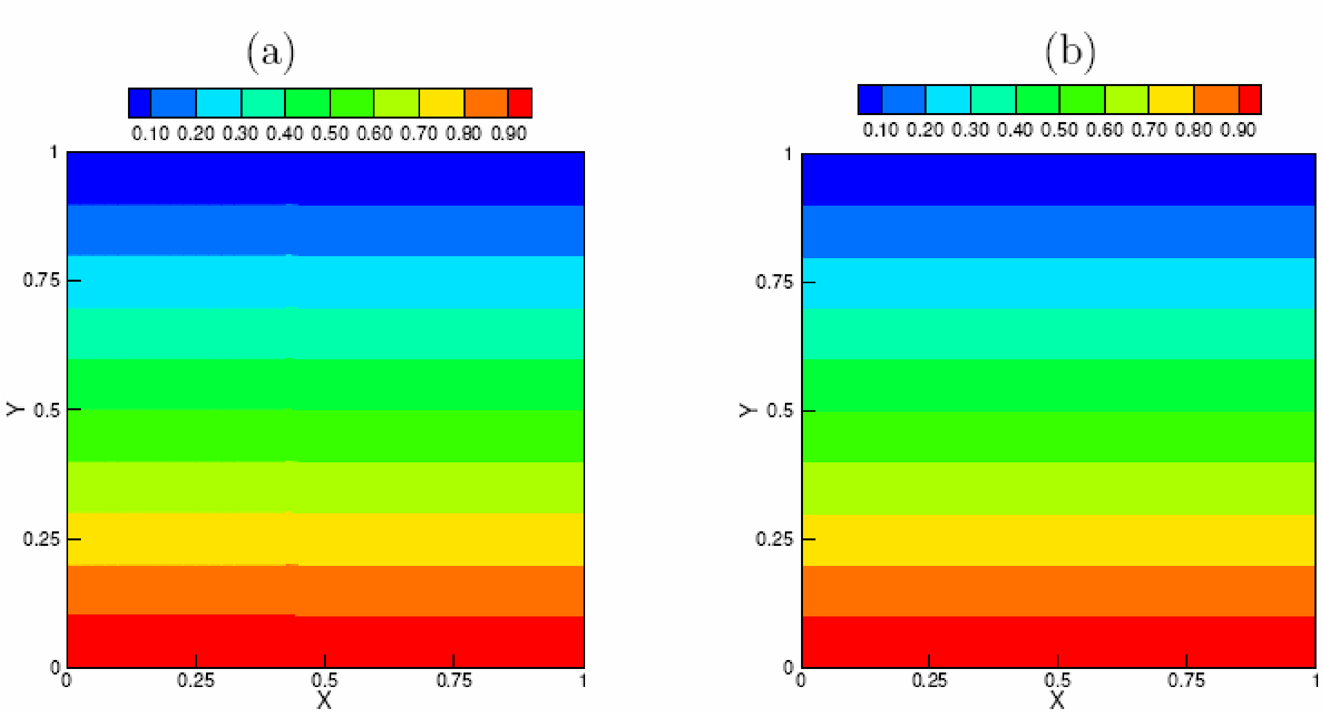


Y-velocity



Asokan and Zabaras, JCP (2005)

PREDICTION BY SUPPORT-SPACE METHOD

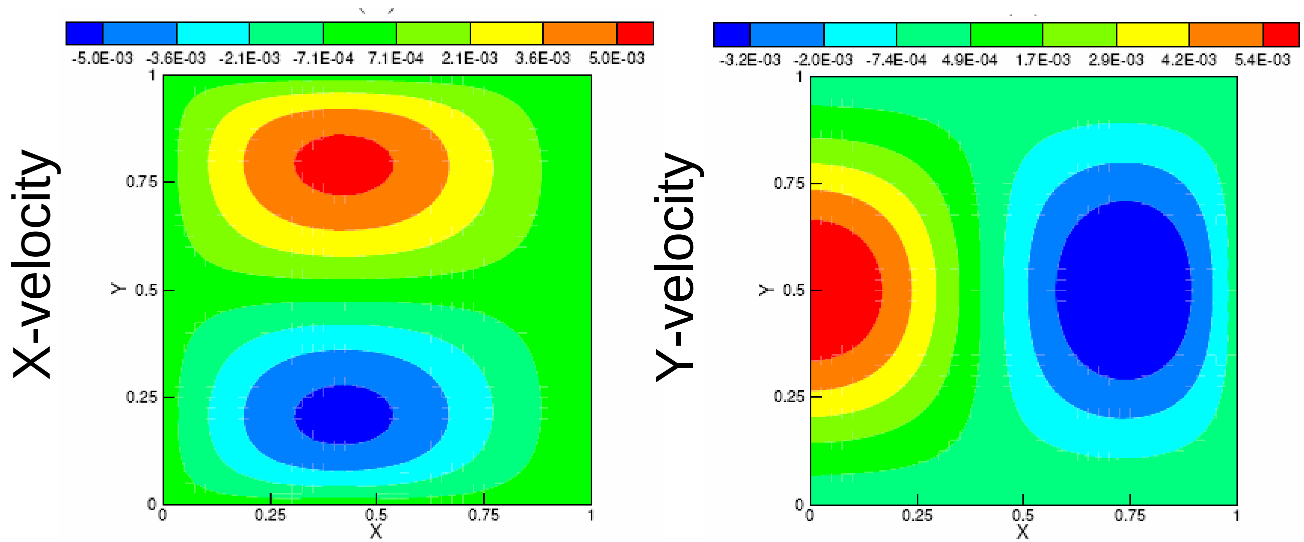


(a) Mean non-dimensional temperature at steady state for deterministic simulation at Ra=1530; (b) Prediction of support-space method for non-dimensional temperature at steady state at Ra = 1530

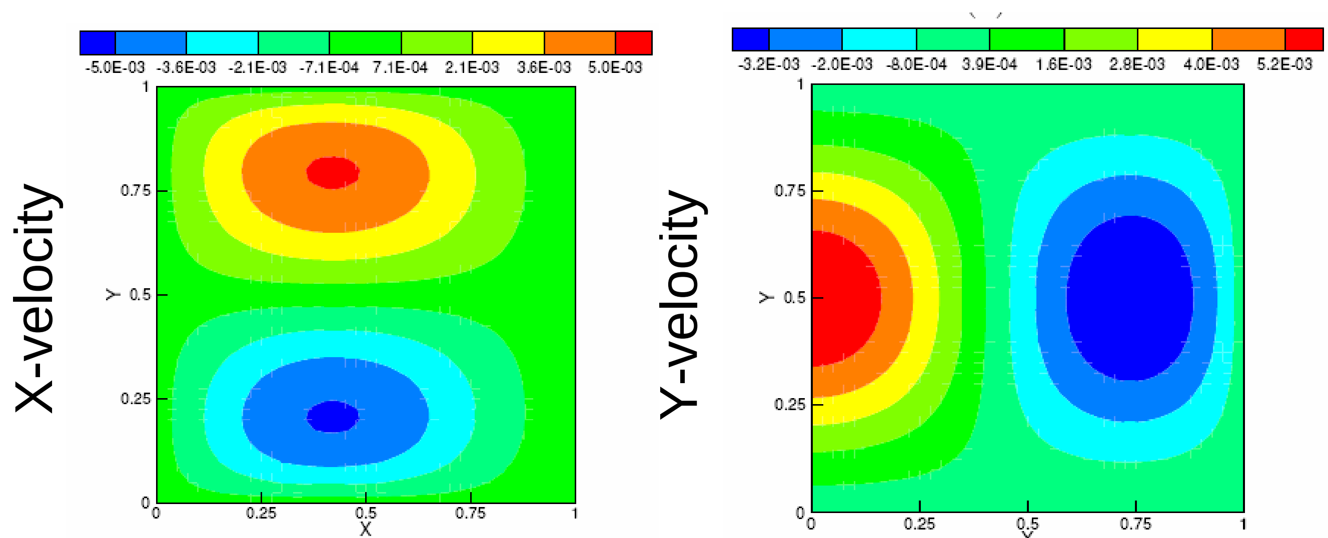
Asokan and Zabaras, JCP (2005)

PREDICTION BY SUPPORT-SPACE METHOD

- ❑ X and Y velocity obtained using a support-space method at Ra = 1870

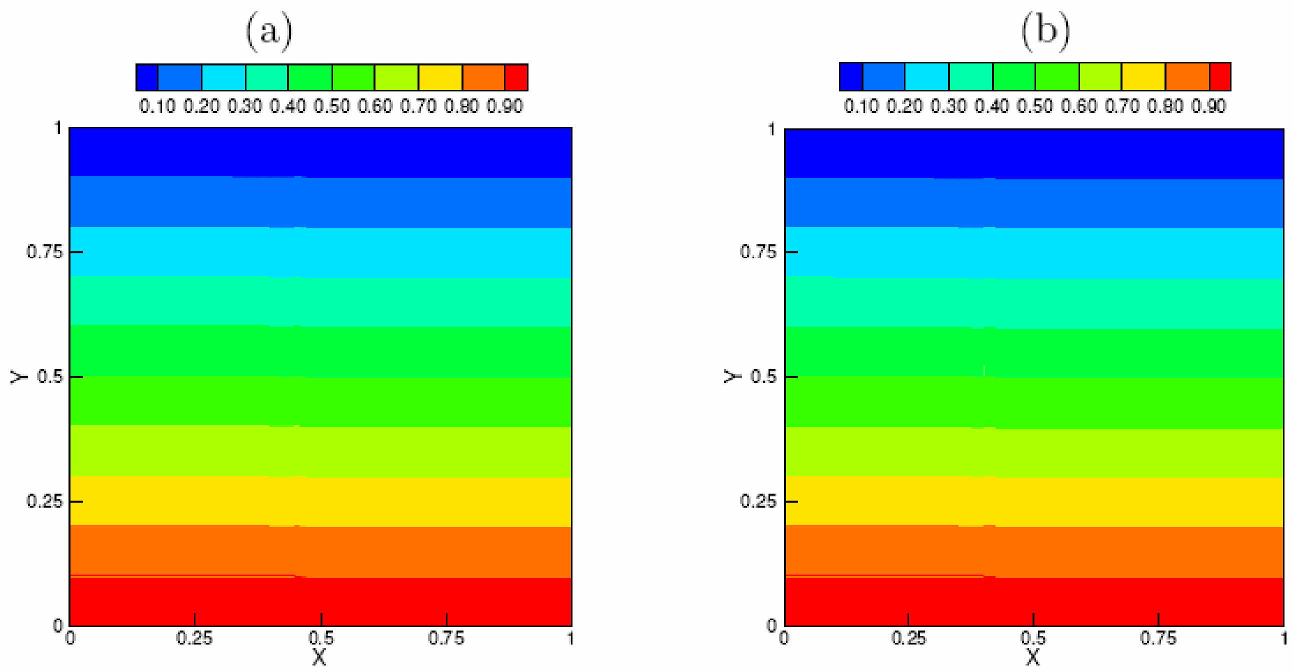


- ❑ Comparative X and Y velocity obtained using a deterministic simulation at Ra = 1870 (the upper limit)



Asokan and Zabaras, JCP (2005)

PREDICTION BY SUPPORT-SPACE METHOD



(a) Mean non-dimensional temperature at steady state for deterministic simulation at Ra=1870; (b) Prediction of support-space method for non-dimensional temperature at steady state at Ra = 1870

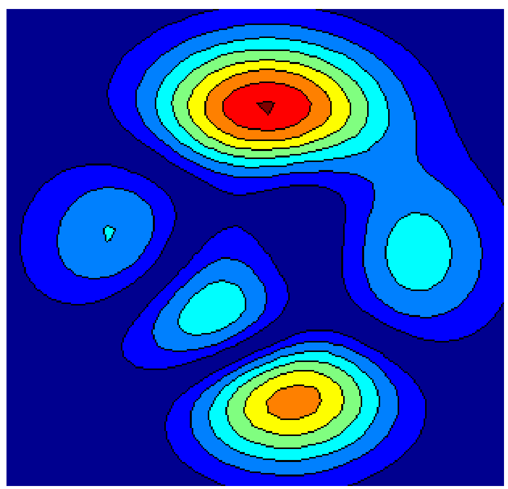
Many potential extensions

ADAPTIVE DISCRETIZATION BASED ON OUTPUT STOCHASTIC FIELD

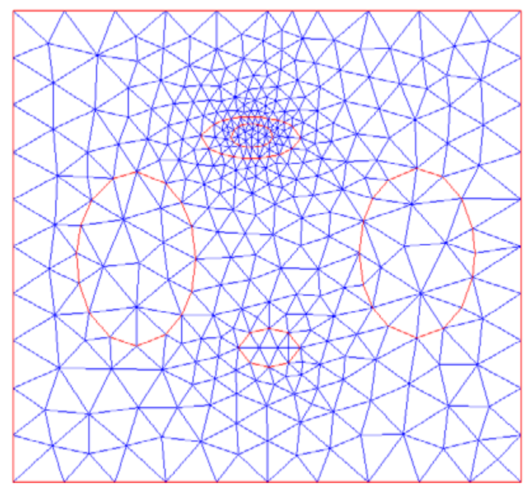
- Refine/Coarsen input support space grid based on output defined control parameter (Gradients, standard deviations etc.)

- Applicable using standard h,p adaptive schemes.

❑ Example of a 2D input and associated support-space



Support-space of input



Importance spaced grid

Many potential extensions

- For support spaces with dimensionality less than 4, standard finite element grids can be used for domain discretization. To extend the approach to domains with dimensionality more than 3, tensor product finite element grid spaces can be considered (I.Babuska et al. 2005)

- We still use the idea of multi-index. First, we introduce a finite dimensional stochastic subspace

$$W^h = \text{span}\{\psi_1(\xi), \psi_2(\xi), \ldots \psi_{N_\varepsilon}(\xi)\}$$

- The construction of W^h can be based on the tensor product of one dimensional basis function $\psi(\xi)$ using multi-index α_m

$$W^h = \text{span}\{\psi_\alpha(\xi) = \prod_{m=1}^{M} \psi_{\alpha_m}(\xi_m) : \psi_{\alpha_m} \in W_m^h\}$$