THE FEM FOR ELLIPTIC PROBLEMS

- Abstract formulation: A unified treatment
- The variational (V_h) and minimization (M_h) problems
- Error estimates
- The energy norm
- Applications to various elliptic boundary value problems
- Supplemental mathematical background
  1. Sobolev spaces $H^k(\Omega), k = 1, 2, \ldots$
  2. The Poincare inequality

ABSTRACT FORMULATION OF THE FEM FOR ELLIPTIC PROBLEMS

- Let V be a Hilbert space with scalar product $(.,.)_V$ and corresponding norm $\|\cdot\|_V$.
- $a(.,.)$ is a bilinear form on $V \times V$ and L a linear form on V such that:
  1. $a(.,.)$ is symmetric
  2. $a(.,.)$ is continuous, i.e., there is constant $\gamma > 0$ such that
     $|a(v,w)| \leq \gamma \|v\|_V \|w\|_V, \qquad \forall\, v, w \in V$
  3. $a(.,.)$ is V-elliptic, i.e., there is a constant $\alpha > 0$ such that
     $a(v,v) \geq \alpha \|v\|_V^2, \qquad \forall\, v \in V$
  4. L is continuous, i.e., there is a constant $\Lambda > 0$ such that
     $|L(v)| \leq \Lambda \|v\|_V, \qquad \forall\, v \in V$

The abstract minimization (M) and variational (V) problems

- The minimization problem (M):

  Find $u \in V$ such that $F(u) = \min_{v \in V} F(v)$,

  where

  $F(v) = \frac{1}{2} a(v, v) - L(v)$

- Variational problem (V):

  Find $u \in V$ such that $a(u, v) = L(v) \qquad \forall\, v \in V$.

Theorem

The problems (M) and (V) are equivalent. There exists a unique solution $u \in V$ of these problems and the following stability estimate holds:

$$\|u\|_V \leq \frac{\Lambda}{\alpha} \quad \text{Stability estimate}$$

Proof of the theorem

1. The existence of a solution follows from the Lax-Milgram theorem.
2. **If $u \in V$ is a solution of (M), then u is a solution of (V).**
   - Let $v \in V$ and $\epsilon \in \Re$ be arbitrary. Then $(u + \epsilon v) \in V$.
   - Since u is a minimum, $F(u) \leq F(u + \epsilon v) \qquad \forall\, \epsilon \in \Re$
   - Let $g(\epsilon) \equiv F(u + \epsilon v), \epsilon \in \Re$. Then $g(0) \leq g(\epsilon)$, $\forall\, \epsilon \in \Re$.
   - Hence $g'(0) = 0$ (if $g'(0)$ exists). Using the symmetry of $a(.,.)$:

$$\begin{aligned} g(\epsilon) &= \frac{1}{2}a(u + \epsilon v, u + \epsilon v) - L(u + \epsilon v) \\ &= \frac{1}{2}a(u,u) + \frac{\epsilon}{2}a(u,v) + \frac{\epsilon}{2}a(v,u) + \frac{\epsilon^2}{2}a(v,v) - L(u) - \epsilon L(v) \\ &= \frac{1}{2}a(u,u) - L(u) + \epsilon a(u,v) - \epsilon L(v) + \frac{\epsilon^2}{2}a\,(v,v) \end{aligned}$$

   - $g'(0) = 0 \Rightarrow g'(0) = a(u,v) - L(v) = 0 \Rightarrow u$ is a solution of (V).

Proof of the theorem

3. **If $u \in V$ is a solution of (V), then u is a solution of (M).**

- Let $v \in V$ and $\epsilon \in \Re$ be arbitrary. Then $(u + \epsilon v) \in V$ with $a(u, v) = L(v), \forall\, v \in V$.
- Using the earlier given expression for $g(\epsilon)$ and the V-elliptic condition for $a(.,.)$, we can write the following:

$$\begin{aligned} F(u + \epsilon v) - F(u) &= \epsilon(a(u, v) - L(v)) + \frac{\epsilon^2}{2} a(v, v) \\ &= \frac{\epsilon^2}{2} a(v, v) \geq \frac{\epsilon^2}{2} \alpha \|v\|_V \geq 0 \end{aligned}$$

  where we used $a(u, v) = L(v)$.
- It follows that u is a solution of (M).

Proof of the theorem

4. **Stability condition**

- To prove the stability condition, we use the continuity of L and ellipticity of $a(.,.)$:

$$\alpha\|u\|_V^2 \le a(u,u) = L(u) \le \Lambda\|u\|_V$$

- For $\|u\|_V \neq 0$, we then conclude:

$$\|u\|_V \le \frac{\Lambda}{\alpha}$$

Proof of the theorem

5. **Uniqueness of solution of** (M) **and** (V)

- Let u_1 and u_2 be two solutions of (V), then:
$$a(u_i, v) = L(v) \qquad \forall\, v \in V, i = 1, 2$$
- By subtraction, we see that: $a(u_1 - u_2, v) = 0, \forall\, v \in V$
- Applying the stability condition for the case of $u = u_1 - u_2$, $L = 0$ (i.e. $\Lambda = 0$), we obtain:
$$\|u_1 - u_2\|_V \leq 0 \Rightarrow u_1 = u_2$$

Important note

Even without the symmetry condition $a(u, v) = a(v, u)$, there exists a unique $u \in V$ such that

$$a(u, v) = L(v) \qquad \forall\, v \in V \qquad\qquad (V)$$

and the stability estimate $\|u\|_V \leq \frac{\Lambda}{\alpha}$ holds. However, in this case, we cannot define an associated minimization problem (M).

Discrete problems (M_h) and (V_h)

- Let V_h be a finite-dimensional subspace of V of dimension M. Let $\{\phi_1, \ldots, \phi_M\}$ be a basis for V_h so that $\phi_i \in V_h$.
- Any $v \in V_h$ has the unique representation

$$v = \sum_{i=1}^{M} \eta_i \phi_i, \text{ where } \eta_i \in \Re$$

We can now formulate the following discrete analogues of the problems (M) and (V):

- Problem (M_h): Find $u_h \in V_h$ such that

$$F(u_h) \leq F(v) \qquad \forall\, v \in V_h \qquad \text{Discrete optimization problem}$$

- Problem (V_h): Find $u_h \in V_h$ such that

$$a(u_h, \phi_j) = L(\phi_j),\ j = 1, \ldots, M. \qquad \text{Discrete variational problem}$$

Matrix form of (V_h)

- Let $u_h = \sum_{i=1}^{M} \xi_i \phi_i, \quad \xi_i \in \Re$. (V_h) takes the matrix form $A\xi = b$:

  $\sum_{i=1}^{M} a(\phi_i, \phi_j)\xi_i = L(\phi_j),\ i, j = 1, \ldots, M,$ with $A_{ji} = a(\phi_i, \phi_j),\ b_j = L(\phi_j)$

- $a(v, v) = a\left(\sum_{i=1}^{M} \eta_i \phi_i, \sum_{j=1}^{M} \eta_j \phi_j\right) = \sum_{i,j=1}^{M} \eta_i a\left(\phi_i, \phi_j\right) \eta_j = \eta \cdot A\eta$

  $L(v) = L\left(\sum_{j=1}^{M} \eta_j \phi_j\right) = \sum_{j=1}^{M} \eta_j L\left(\phi_j\right) = b \cdot \eta,\ \forall\ v\ \in\ V_h,$

  where $\cdot$ is the dot product in $\Re^M$.

- Problem (M_h): $\frac{1}{2}\xi \cdot A\xi - b \cdot \xi = \min_{\eta \in \Re^M} \left[\frac{1}{2}\eta \cdot A\eta - b \cdot \eta\right]$

- Properties of matrix A:
  - $\eta \cdot A\eta = a(v, v) \geq \alpha \|v\|_V^2 > 0$ (for $v \neq 0$) $\Rightarrow \eta \cdot A\eta > 0$, for $\eta \neq 0 \Rightarrow$ A is positive definite.
  - $a(\phi_i, \phi_j) = a(\phi_j, \phi_i) \Rightarrow A$ is symmetric.
  - There is a unique solution ξ of the system $A\xi = b$.

Stability condition for the solution of problems (V_h) and (M_h)

There exists a unique solution $u_h \in V_h$ to the equivalent problems (V_h) and (M_h). Further, the following stability estimate holds:

$$\|u_h\|_V \leq \frac{\Lambda}{\alpha}$$

- The stability estimate follows by choosing $v = u_h$ in $a(u_h, v_h) = L(v_h)$ and using the V-ellipticity condition for $a(.,.)$ and the L-continuity condition, i.e.

$$\alpha\|u_h\|_V^2 \leq a(u_h, u_h) = L(u_h) \leq \Lambda\|u_h\|_V$$

- From the above equation for $\|u_h\|_V \neq 0$, the stability condition is derived.

The above stability estimate can be viewed as the theoretical basis for the success of the FE method

Error estimate

Theorem

Let $u \in V$ be the solution of (V) and $u_h \in V_h$ that of (V_h) where $V_h \subset V$.
Then:

$$\|u - u_h\|_V \le \frac{\gamma}{\alpha}\|u - v\|_V \qquad \forall\, v \in V_h \qquad \text{Error estimate}$$

Recall that:

- γ appears in the $a(.,.)$-continuity condition:
  $|a(v,w)| \le \gamma\|v\|_V\|w\|_V, \qquad \forall\, v, w \in V$
- α appears in the ellipticity condition for $a(.,.)$:
  $a(v,v) \ge \alpha\|v\|_V^2, \qquad \forall\, v \in V$

Proof of the error estimate

- We can easily show that: $a(u-u_h, w) = 0 \qquad \forall\, w \in V_h.$
- For an arbitrary $v \in V_h$, define $w = u_h - v$. Using the $V-$ellipticity and continuity conditions of $a(.,.)$ and that $a(u-u_h, w) = 0$, we obtain:

$$\begin{aligned}\alpha\|u-u_h\|_V^2 &\overbrace{\leq}^{\text{ellipticity}} a(u-u_h, u-u_h) + \overbrace{a(u-u_h, w)}^{=\,0} \\ &= a(u-u_h, u-u_h+w) = a(u-u_h, u-v) \\ &\overbrace{\leq}^{\text{continuity}} \gamma\|u-u_h\|_V\|u-v\|_V\end{aligned}$$

- Division by $\|u-u_h\|_V \neq 0$ provides the required error estimate.
- Can choose $v = \pi_h u$ where $\pi_h u \in V_h$ is a suitable interpolant of u. We would then like to calculate the *interpolation error* $\|u-\pi_h u\|_V$.
- The FEM error estimate can now take the following form:

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha}\|u-\pi_h u\|_V \qquad \text{Error estimate}$$

The energy norm: $\|v\|_a = \sqrt{a(v,v)}, v \in V$

- Define the energy norm $\|.\|_a$ as follows: $\|v\|_a^2 = a(v,v), v \in V$.
- $\|.\|_a$ is equivalent to $\|.\|_V$:
  $c\|v\|_V \leq \|v\|_a \leq C\|v\|_V, \ \forall\ v \in V$, with $c = \sqrt{\alpha}$ and $C = \sqrt{\gamma}$
- The corresponding inner product is defined as: $(v,w)_a = a(v,w)$.
- We showed that u_h is the projection of u onto V_h, i.e.:

$$(u - u_h, w)_a = 0 \qquad \forall\ w \in V_h$$

- Using $(u-u_h,w)_a = 0, \forall\ w \in V_h$ and the Cauchy inequality, we obtain:

$$\begin{aligned} \|u - u_h\|_a^2 &= (u-u_h, u-u_h)_a = (u-u_h, u-u_h)_a + (u-u_h, w)_a \\ &= (u-u_h, u-u_h+w)_a = (u-u_h, u-v)_a \\ &\leq \|u-u_h\|_a \|u-v\|_a \rightarrow \|u-u_h\|_a \leq \|u-v\|_a, \forall\ v \in V_h \end{aligned}$$

  i.e. u_h is a best approximation of u in the energy norm.

Example 1: $-\Delta u + u = f$ in Ω, $\frac{\partial u}{\partial n} = 0$ on Γ and $f \in L_2(\Omega)$

- $V = H^1(\Omega)$ and the following variational problem is obtained:

$$\left.\begin{array}{l} a(v,w) = \int_\Omega [\nabla v \cdot \nabla w + vw]\, dx \\ L(v) = \int_\Omega fv dx \end{array}\right\}$$

- $a(.,.)$ is a symmetric bilinear form on $V \times V$.
- L is a linear form.
- $a(v,v) = \|v\|^2_{H^1(\Omega)} \Rightarrow a(.,.)$ is $V-$elliptic and using Cauchy's inequality we obtain:

$$a(v,w) \leq a(v,v)^{\frac{1}{2}} a(w,w)^{\frac{1}{2}} = \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \Rightarrow a \text{ is continuous}$$

- The continuity of L is shown using the Cauchy inequality in L_2:

$$|L(v)| \leq \left|\int_\Omega fv dx\right| \leq \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)} \|v\|_{H^1(\Omega)}$$

Example 2: $-u'' = f$ on $I = (0, 1)$, $u(0) = u(1) = 0$ and $f \in L_2(I)$

- $V = H_0^1(I)$ and the following variational problem is obtained:
$$a(v, w) = \int_I v'w'dx, \quad L(v) = \int_I fvdx$$
- $a(.,.)$ is obviously symmetric and bilinear and L is linear. The continuity of L is derived as in Example 1.
- The continuity of $a(.,.)$ is shown as follows:
$$|a(v, w)| \leq \|v'\|_{L_2(I)} \|w'\|_{L_2(I)} \leq \|v\|_{H_0^1(I)} \|w\|_{H_0^1(I)}$$
- The $V-$elliptic condition for $a(.,.)$ can be shown using the fact that
$$\int_I v^2dx \leq \int_I (v')^2dx \qquad \forall\, v \in H_0^1(I)$$
- $a(v, v) = \int_I (v')^2dx \geq \frac{1}{2}\left(\int_I v^2dx + \int_I (v')^2dx\right) = \frac{1}{2}\|v\|_{H_0^1(I)}, \ \forall\, v \in H_0^1(I).$
- If V_h consists of piecewise linear functions on I and u is smooth enough:
$$\|u - u_h\|_{H^1(\Omega)} \leq Ch$$

Proof of $\int_I v^2 dx \leq \int_I (v')^2 dx, \; \forall \; v \in H_0^1(I)$

- Using $v(0) = 0$, we can write: $v(x) = v(0) + \int_0^x v'(y)dy = \int_0^x v'(y)dy$
- Using Cauchy's inequality:

$$\begin{aligned} |v(x)| &= |\int_0^x v'(y)dy| \leq \int_0^1 |v'(y)|dy \\ &\leq (\int_0^1 1^2 dy)^{\frac{1}{2}} \; (\int_0^1 (v'(y))^2 dy)^{\frac{1}{2}} = (\int_0^1 (v'(y))^2 dy)^{\frac{1}{2}} \end{aligned}$$

- Taking the square and then integrating the last equation from 0 to 1:

$$\int_0^1 v^2(x)dx \leq \int_0^1 (v'(y))^2 dy$$

- The above proof needs the boundary condition $v(0) = 0$ (e.g. take $v = 1$ to see that the inequality does not stand!).
- In order to control the norm of the function v by the norm of the derivative v' we need a "fixed point" to start from.

Example 3: $-\Delta u = f$ in $\Omega \subset \Re^2$, $u = 0$ on Γ, with $f \in L_2(\Omega)$

- Here, $V = H_0^1(\Omega)$, with $a(v,w) = \int_\Omega \nabla v \cdot \nabla w dx$, $L(v) = \int_\Omega f v dx$
- $a(.,.)$ is symmetric & bilinear and L is linear. For the continuity of a:

$$|a(v,w)| \leq \||\nabla v|\|_{L_2(\Omega)} \||\nabla w|\|_{L_2(\Omega)} \leq \|v\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)}$$

- The $V-$ellipticity can be shown using Poincare's inequality:

$$\int_\Omega v^2 dx \leq c \int_\Omega |\nabla v|^2 dx \; \forall \; v \in H_0^1(\Omega)$$

where c is a constant. Indeed, we can write:

$$a(v,v) \equiv \int_\Omega |\nabla v|^2 dx \geq \frac{1}{c+1} \left(\int_\Omega \left(v^2 + |\nabla v|^2 \right) dx \right) \equiv \frac{1}{c+1} \|v\|^2_{H^1(\Omega)}$$

- For piecewise continuous functions in Ω and u sufficiently smooth:

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch$$

Example 4: $\frac{d^4u}{dx^4} = f,\ x \in I = (0,1),$
$u(0) = u'(0) = u(1) = u'(1) = 0$ with $f \in L_2(I)$

- Introduce: $H^2(I) = \{v \in L_2(I) : v', v'' \in L_2(I)\}$. Then $V = H_0^2(I)$, where:

$$H_0^2(I) = \{v \in H^2(I) : v(0) = v'(0) = v(1) = v'(1) = 0\}$$

- Find $u \in V$ such that $a(u,v) = L(v)\ \forall\ v \in V$, where

$$a(v,w) = \int_I v''w''dx, \quad L(v) = \int_I fv dx$$

- The symmetry of $a(.,.)$ and continuity of $a(.,.)$ and L are easy to show.
- Using twice the Poincare inequality and $v(0) = v'(0) = 0$, we can write:

$$\int_I v^2 dx \leq \int_I (v')^2 dx \leq \int_I (v'')^2 dx,\ \forall\ v \in\ H_0^2(I) \Rightarrow$$

$$\|v\|^2_{H^2(I)} = \int_I \{v^2 + (v')^2 + (v'')^2\} dx \leq 3 \int_I (v'')^2 dx \equiv 3a(v,v)$$

(ellipticity condition for a with $\alpha = 1/3$)

Sobolev spaces $H^k(\Omega), k = 1, 2, \ldots$

- Let $D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, $|\alpha| = \alpha_1 + \alpha_2, \alpha_i$ non-negative natural number
- We now define for $k = 1, 2, \ldots,$

$$H^k(\Omega) = \{v \in L_2(\Omega) : D^\alpha v \in L_2(\Omega), |\alpha| \leq k\} \qquad \text{Sobolev space}$$

with norm

$$\|v\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha v|^2 \, dx \right)^{\frac{1}{2}} \qquad \text{Sobolev space norm}$$

- The space $H^k(\Omega)$ consists of all functions v on Ω that, together with the partial derivatives $D^\alpha v$ of order $|\alpha|$ at most k, belong to $L_2(\Omega)$.
- The space $H^k(\Omega)$ is a Hilbert space.

Example 5: The biharmonic problem

$$\Delta\Delta u = f \text{ in } \Omega, \;\; u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma$$

- Physical models: Clamped plate under transverse load,

  Stokes equations in fluid mechanics, etc.
- Here $V = H_0^2(\Omega) = \left\{ v \in H^2(\Omega) : v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \right\}$.
- Use: $\int_\Omega (\Delta\Delta u)\, v dx = \int_\Gamma \frac{\partial \Delta u}{\partial n}\, v ds - \int_\Gamma \Delta u \frac{\partial v}{\partial n} ds + \int_\Omega \Delta u \Delta v dx = \int_\Omega \Delta u \Delta v dx$.
- (V): Find $u \in V$ such that: $a(u,v) = L(v)$, with:

  $$a(u,v) = \int_\Omega \Delta u \Delta v dx \text{ and } L(v) = \int_\Omega f v dx$$
- Easy to show the symmetry of $a(.,.)$ and continuity of $a(.,.)$ and L.
- To show the V-ellipticity of $a(.,.)$ need to first show the Poincare inequality:

  $$\|v\|^2_{H^2(\Omega)} \leq c \int_\Omega (\Delta v)^2 dx, \; \forall \; v \in H_0^2(\Omega), \text{ for some constant } c$$

  (see derivation at the end of the lecture).

Example 6: A convection-diffusion problem

$-\mu\Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f$ in Ω, $u = 0$ on Γ, μ, β constants with $\mu > 0$

- Convection in the direction $\beta = (\beta_1, \beta_2)$. Assume $|\beta|/\mu$ is small. We next simplify by taking $\mu = 1$.

- $v \in V = H_0^1(\Omega)$ and (V): Find $u \in V$: $a(u, v) = L(v)\ \forall\ v \in V$, where:
$$a(v, w) = \int_\Omega \{\nabla v \cdot \nabla w + \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2} + v\right) w\} dx, \quad L(v) = \int_\Omega f v dx$$

- Note that: $\int_\Omega \{\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}\}\ v\ dx = \int_\Gamma \{v^2(\beta_1 n_1 + \beta_2 n_2\} ds - \int_\Omega \{\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}\}\ v\ dx \Rightarrow \int_\Omega \{\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}\}\ v\ dx = 0$.

- Thus: $a(v, v) = \int_\Omega \left[|\nabla v|^2 + v^2\right] dx = \|v\|^2_{H^1(\Omega)} \Rightarrow a(.,.)$ is V-elliptic.

- The stiffness matrix A is not symmetric. As noted earlier (p. 7), there still exists a unique solution $u_h \in V_h$ to (V_h): $a(u_h, v) = L(v)\ \forall\ v \in V_h$.

- The following error estimate is obtained ($\alpha = 1$):
$$\|u - u_h\|_{H^1(\Omega)} \le \gamma \|u - v\|_{H^1(\Omega)} \quad \forall\ v\ \in V_h$$

Example 7: A heat conduction problem:

$$q_i = -k_i(x)\frac{\partial u}{\partial x_i} \text{ in } \Omega \text{ (no-sum on } i)$$

$$\text{with div } q = f \text{ in } \Omega,\ u = 0 \text{ on } \Gamma_1,\ q \cdot n = -g \text{ on } \Gamma_2$$

- With the k_i non-constant, this is an example of a partial differential equation with variable coefficients.
- Here $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$ with $a(u, v) = L(v), \forall\, v \in V$, where:

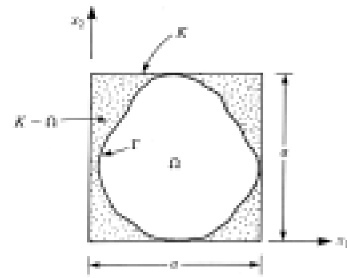
$$a(v, w) = \int_\Omega \sum_{i=1}^{3} k_i(x)\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}dx,$$
$$L(v) = \int_\Omega fv dx + \int_{\Gamma_2} gv ds.$$

- The general conditions are satisfied under the following hypothesis: There are positive constants c and C such that

$$c \leq k_i(x) \leq C,\ x \in \Omega,\ i = 1, 2, 3,$$

$f \in L_2(\Omega)$, $g \in L_2(\Gamma_2)$ and the area of Γ_1 is positive.

The Poincare inequality: $\int_\Omega v^2 dx \leq c(\Omega) \int_\Omega |\nabla v|^2 dx, \ \forall \ v \in \ H_0^1(\Omega)$



- Let Ω be an open bounded region. The region $\bar{\Omega} = \Omega + \Gamma$ can be enclosed in a square region K: $K = \{(x_1, x_2) : x_1 \ \in \ [0, a], \ x_2 \ \in \ [0, a]\}$.
- $v(x)$ is extended onto K be setting $v = 0$ on $K - \Omega$. Hence, v is continuously differentiable in K and vanishes on the boundary of K.
- Using: $v(x_1, x_2) = \int_0^{x_1} \frac{\partial v}{\partial \xi}(\xi, x_2) d\xi$ and the Cauchy inequality, we obtain:

$$|v(x_1, x_2)|^2 = |\int_0^{x_1} \frac{\partial v}{\partial \xi}(\xi, x_2) d\xi|^2 \leq ((\int_0^{x_1} 1^2 d\xi)^{1/2} (\int_0^{x_1} |\frac{\partial v}{\partial \xi}|^2 d\xi)^{1/2})^2$$
$$\leq x_1 \int_0^{x_1} |\frac{\partial v}{\partial \xi}|^2 d\xi \leq a \int_0^{a} |\frac{\partial v}{\partial \xi}|^2 d\xi$$

The Poincare inequality

- Integrating with respect to x_1 and x_2 over K, we obtain:

$$\int_0^a\int_0^a |v(x_1,x_2)|^2 dx_1 dx_2 \;\leq\; a\int_0^a\int_0^a\left(\int_0^a \left|\frac{\partial v(\xi,x_2)}{\partial \xi}\right|^2 d\xi\right) dx_1 dx_2$$
$$= a^2 \int_0^a\int_0^a \left|\frac{\partial v}{\partial \xi}\right|^2 d\xi\, dx_2$$
$$= a^2 \int_K \left|\frac{\partial v}{\partial x_1}\right|^2 dx_1 dx_2$$

- Similarly we obtain: $\int_0^a\int_0^a |v(x_1,x_2)|^2 dx_1 dx_2 \leq a^2 \int_K |\frac{\partial v}{\partial x_2}|^2 dx_1 dx_2$.
- From the above two inequalities and using $v = 0$ on $K - \Omega$, we obtain:
$$\int_\Omega |v(x_1,x_2)|^2 dx_1 dx_2 \leq \frac{a^2}{2}\int_\Omega \left(|\frac{\partial v}{\partial x_1}|^2 + |\frac{\partial v}{\partial x_2}|^2\right) dx_1 dx_2$$
- We conclude that: $\int_\Omega v^2 dx \;\leq\; c(\Omega)\int_\Omega |\nabla v|^2 dx,\ \forall\ v \in\ H_0^1(\Omega)$, where $c(\Omega) > 0$ is a constant that depends on Ω (c dictates the value of a only).

The Poincare inequality: $\|v\|^2_{H^2(\Omega)} \leq C(\Omega) \int_\Omega (\Delta v)^2 dx, \ \forall \ v \in \ H^2_0(\Omega)$

- From $v = 0$ on $\Gamma \Rightarrow \frac{\partial v}{\partial s}$ on Γ. With $\frac{\partial v}{\partial n} = 0$ on $\Gamma \Rightarrow \frac{\partial v}{\partial x_1} = \frac{\partial v}{\partial x_2} = 0$ on Γ.
- Using these boundary conditions and integrating by parts first in x_1 and then in x_2, one can easily show that:

$$\int_\Omega \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} \, dx_1 \, dx_2 = \int_\Omega \left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2 dx_1 \, dx_2, \quad \forall \ v \ \in \ H^2_0(\Omega) \qquad \text{(a)}$$

- Using the earlier Poincare inequality for functions in $H^1_0(\Omega)$, we obtain:

$$\int_\Omega \left(\frac{\partial v}{\partial x_1}\right)^2 dx_1 \, dx_2 \leq c(\Omega) \ \int_\Omega \left\{\left(\frac{\partial^2 v}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2\right\} dx_1 \, dx_2 \qquad \text{(b)}$$

$$\int_\Omega \left(\frac{\partial v}{\partial x_2}\right)^2 dx_1 \, dx_2 \leq c(\Omega) \ \int_\Omega \left\{\left(\frac{\partial^2 v}{\partial x_2^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2\right\} dx_1 \, dx_2 \qquad \text{(c)}$$

$$\int_\Omega v^2 \, dx_1 \, dx_2 \leq c(\Omega) \ \int_\Omega \left\{\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2\right\} dx_1 \, dx_2 \qquad \text{(d)}$$

- From (b)+(c) and (a): $\int_\Omega |\nabla v|^2 \, dx_1 \, dx_2 \leq c(\Omega) \ \int_\Omega (\Delta v)^2 dx_1 \, dx_2$ (e)
- From (d) and (e): $\int_\Omega v^2 \, dx_1 \, dx_2 \leq c^2(\Omega) \ \int_\Omega (\Delta v)^2 dx_1 \, dx_2$ (f)
- $\|v\|^2_{H^2(\Omega)} = \int_\Omega \left\{v^2 + (\nabla v)^2 + \left(\frac{\partial^2 v}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2\right\} dx_1 \, dx_2 \Rightarrow$
  $\|v\|^2_{H^2(\Omega)} = \int_\Omega \{v^2 + (\nabla v)^2 + (\Delta v)^2\} dx_1 \, dx_2 \leq \underbrace{(c^2 + c + 1)}_{C} \int_\Omega (\Delta v)^2 dx_1 \, dx_2$

The semi-norm $|v|_1$: $|v|_1 = \int_\Omega [|\frac{\partial v}{\partial x_1}|^2 + |\frac{\partial v}{\partial x_1}|^2] dx_1 dx_2$

$$c_2 \|v\|_1 \leq |v|_1 \leq c_1 \|v\|_1, \quad \forall\, v \in \; H_0^1(\Omega)$$

- Based on the first Poincare inequality, we can write that:

$$\int_\Omega v^2 dx_1 dx_2 \leq C^2(\Omega) \int_\Omega (|\frac{\partial v}{\partial x_1}|^2 + |\frac{\partial v}{\partial x_2}|^2) dx_1 dx_2$$

- Define $c_2^2 = \frac{1}{2}\, min\{1, \frac{1}{C^2}\}$.
- We thus can conclude:

$$c_2^2 \int_\Omega [v^2 dx + |\frac{\partial v}{\partial x_1}|^2 + |\frac{\partial v}{\partial x_1}|^2] dx_1 dx_2 \leq \int_\Omega (|\frac{\partial v}{\partial x_1}|^2 + |\frac{\partial v}{\partial x_2}|^2) dx_1 dx_2$$

- It is also clear that (e.g. for $c_1 = 1$):

$$|v|_1 \;\leq\; c_1 \; \|v\|_1, \quad \forall\, v \in \; H^1(\Omega)$$

- We can now conclude that:

$$c_2 \|v\|_1 \leq |v|_1 \leq c_1 \|v\|_1, \quad \forall\, v \in \; H_0^1(\Omega)$$

- For $v \; \in H_0^1(\Omega)$, the seminorm $|v|_1$ is thus equivalent to the norm $\|v\|_1$.