MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

Cornell University, Fall 2009

Nicholas Zabaras
Materials Process Design and Control Laboratory
Sibley School of Mechanical and Aerospace Engineering
101 Rhodes Hall
Cornell University
Ithaca, NY 14853-3801

Weak Form

- One of the key elements of the FE analysis is the derivation of the weak (integral) form of the governing differential equations (strong form).

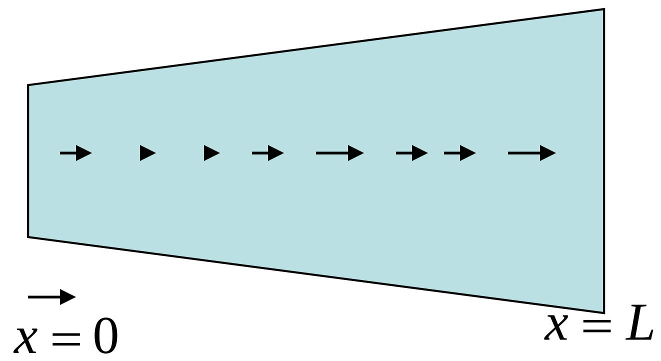
- This is the most critical step in the FEM.

- The process is common in all FEM procedures.

- Introduction of interpolation functions in the weak form will lead to the discretized FE equations.

- In this lecture, we work with 1D boundary value problems governed by ordinary differential eqs. (ODEs).

Strong form: Axial loading of an elastic bar



- Consider an elastic bar loaded as shown above. We introduce the following notation:

$$u(x) \quad \text{Displacement at point } x$$

$$\sigma(x) \quad \text{Stress (force/area) at point } x$$

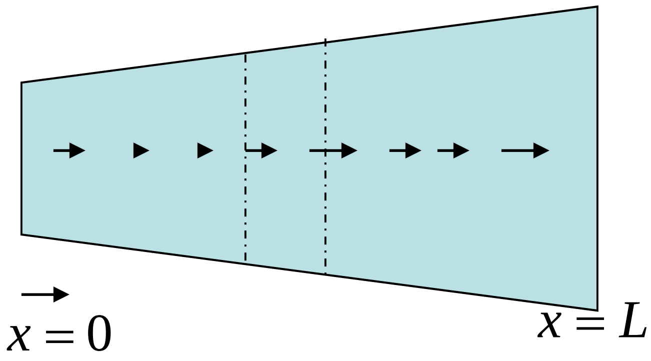
$$\varepsilon(x) = \frac{du}{dx} \quad \text{Strain at point } x$$

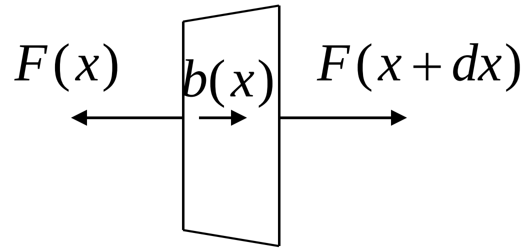
$$F(x) \quad \text{Internal force at point } x$$

$$A(x) \quad \text{Cross sectional area at point } x$$

$$b(x) \quad \text{Distributed load (force/length) at point } x$$

Strong form: Axial loading of an elastic bar





$$-F(x)+b(x+\frac{dx}{2})dx+F(x+dx)=0$$

$$\frac{F(x+dx)-F(x)}{dx}+b(x+\frac{dx}{2})=0,\ take\ dx\rightarrow 0,$$

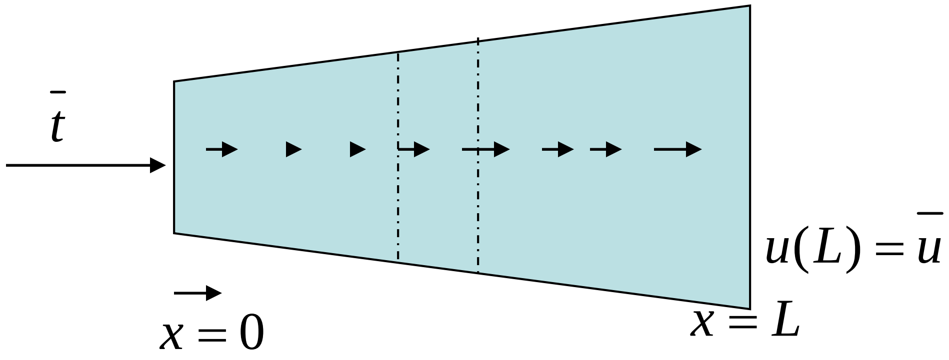
$$\frac{dF}{dx}+b=0$$

- We apply force balance on a differential element.
- We will use the displacement $u(x)$ as the main unknown.
- Using $\sigma=E\varepsilon\ (Hooke's\ law-constitutive\ equation)$, we can write:

$F=\sigma A=E\varepsilon A=EA\frac{du}{dx}$, and the differential equation becomes:

$$\frac{d}{dx}(EA\frac{du}{dx})+b=0,\ 0<x<L$$

Strong form: Axial loading of an elastic bar



Sign convention

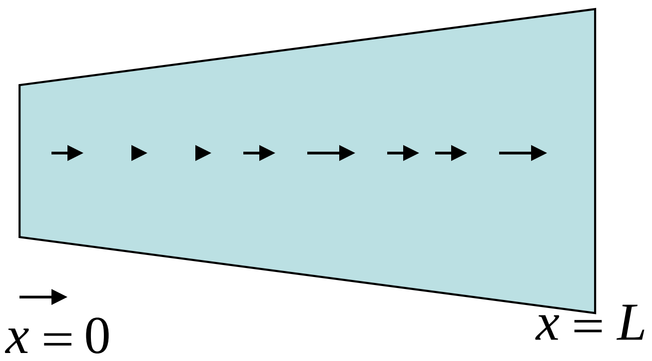
- **The prescribed traction is taken as positive when is acting in the positive direction regardless on which face (left or right) is acting.**
- **The stress is positive in tension & negative in compression.**

- To complete our problem definition, we need boundary conditions – in each end we need to prescribe either the displacement or the traction (force/area).
- For demonstration, let us take prescribed displacement $\bar{u}$ at $x=L$ and prescribed traction $\bar{t}>0$ at $x=0$.
- So **the complete strong form of our BVP** is the following:

$$\frac{d}{dx}(EA\frac{du}{dx})+b=0,\, 0<x<L$$

$$u(L)=\bar{u} \qquad \sigma(0)=E\frac{du}{dx}(0)=-\bar{t}$$

Heat conduction in one-dimension



- Consider heat conduction through a bar as shown above. We introduce the following notation:

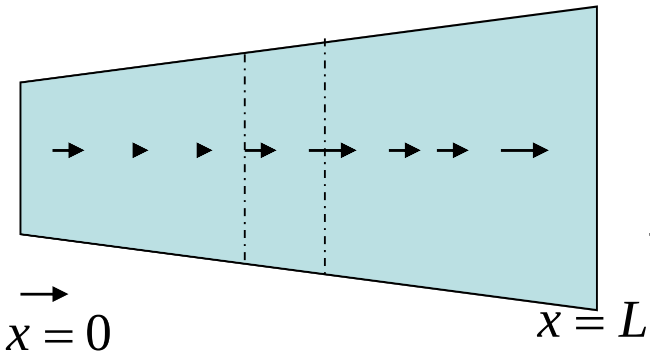
$$T(x) \quad Temperature\ at\ point\ x$$

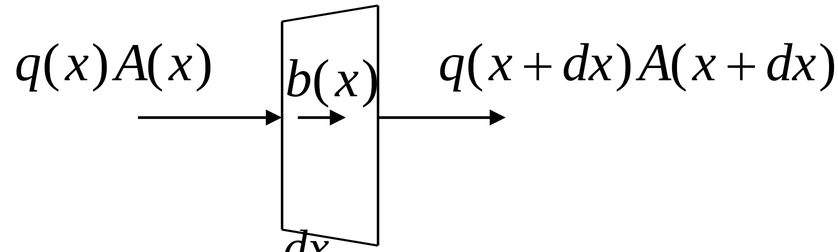
$$q(x) \quad Heat\ flux\ (energy/(area \times time))\ at\ point\ x$$

$$A(x) \quad Cross\ sectional\ area\ at\ point\ x$$

$$b(x) \quad Heat\ source\ (energy/length)\ at\ point\ x$$

Strong form: Heat conduction in a bar





$$-\underbrace{q(x)A(x)}_{\text{heat flow in the CV}} - \underbrace{b(x+\frac{dx}{2})}_{\text{heat generated in the CV}} dx + \underbrace{q(x+dx)A(x+dx)}_{\text{heat flow out of the CV}} = 0$$

$$\frac{q(x+dx)A(x+dx)-q(x)A(x)}{dx} - b(x+\frac{dx}{2}) = 0, \textit{ take } dx \to 0,$$

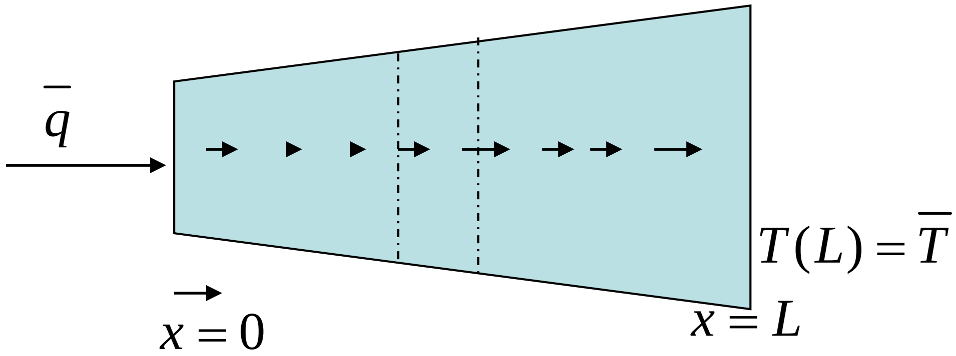
$$\frac{d\left(qA\right)}{dx} = b$$

- We apply energy balance on a differential element.
- We will use the temperature $T(x)$ as the main unknown.

Substituting, $q = -k\frac{dT}{dx}$ $(Fourier\ law - constitutive\ equation)$, leads to:

$$-\frac{d}{dx}(kA\frac{dT}{dx}) = b, \ 0 < x < L$$

Strong form: Heat conduction in a bar



Sign convention

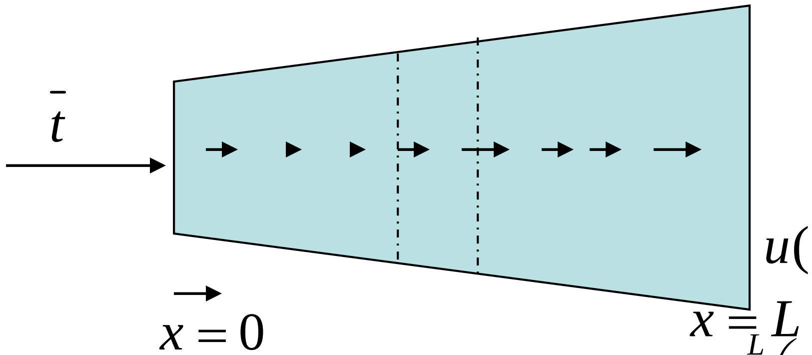
- **The prescribed flux is taken as positive if heat flows out of the bar.**
- **The heat flux q(0) as defined enters the system (i.e. is negative)**

- To complete our problem definition, we need boundary conditions – in each end we need to prescribe either the temperature or the heat flux.
- For demonstration, let us take prescribed temperature $\bar{T}$ at $x=L$ and prescribed heat flux $\bar{q}>0$ at $x=0$.
- So **the complete strong form of our BVP** is the following:

$$\frac{d}{dx}(kA\frac{dT}{dx})+b=0,\, 0<x<L$$

$$T(L)=\bar{T} \qquad q(0)=-k\frac{dT}{dx}(0)=-\bar{q}$$

Weak form: Axial loading of an elastic bar



$$u(L) = \bar{u}$$

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + b = 0,\; 0 < x < L$$

$$u(L) = \bar{u} \qquad \sigma(0) = E\frac{du}{dx}(0) = -\bar{t}$$

$$\int_0^L \left(\frac{d}{dx}\left(EA\frac{du}{dx}\right) + b\right) w(x)\,dx = 0,\; \forall w(x) \text{ in } 0 < x < L$$

$$u(L) = \bar{u}$$

$$\left(wA\left(\underbrace{E\frac{du}{dx}(0)}_{\sigma} + \bar{t} \right)\right)_{x=0} = 0 \;\forall w \text{ on } x = 0$$

- To derive the weak form, multiply the ODE by an arbitrary weight function *w(x)* and integrate on the domain.
- Similarly for the natural BC.
- These two equs. to be equivalent with those in the strong form, need to be true for ALL *w*. We assume (we will come to this shortly) that *w(L)=0*.

Weak form: Axial loading of an elastic bar

Compute *u(x):* $$\int_0^L \left(\frac{d}{dx}(EA\frac{du}{dx}) + b \right) w(x)dx = 0, \forall w(x) \; in \; 0 < x < L$$

$$u(L) = \bar{u}$$

$$\left(wA\left(E\frac{du}{dx}(0) + \bar{t} \right) \right)_{x=0} = 0 \; \forall w \; in \; 0 \le x \le L$$

- Integration by parts of the 1st equation gives:

$$EA\frac{du}{dx} w \Big|_{x=0}^{x=L} - \int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx + \int_0^L bwdx = 0, \forall w(x) \; in \; 0 < x < L, \; with \; w(L) = 0$$

- Using the weak form of the traction BC and that $w(L) = 0$, we can simplify:

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = wA\bar{t}\,\Big|_{x=0} + \int_0^L bwdx, \forall w(x) \; in \; 0 < x < L, \; with \; w(L) = 0$$

Weak problem statement

- Find *u(x)* among the smooth functions with $u(L)=\bar{u}$ such that

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = wA\bar{t}\,|_{x=0} + \int_0^L bwdx,\ \forall w(x)\ in\ 0<x<L,\ with\ w(L)=0$$

- The integration by parts leads to a symmetric form in u and w – this leads to a symmetric stiffness after finite element discretization is introduced.
- The weak statement is EQUIVALENT to the strong form. In other words, not only if *u(x)* satisfies the strong form the above is true, but also if *u(x)* satisfies the weak form then it is also the solution of the strong form of our problem.
- Smoothness in *u* and *w* is needed for the integrals in the weak form to exist (**we need H^1 functions u and w that have 1st derivative that is squared integrable**)!

1^{st} derivative squared integrable functions

- What smoothness conditions on *u* and *w* we need for the integrals in the weak form to exist?

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = wA\bar{t}\,|_{x=0} + \int_0^L bwdx,\ \forall w(x)\ in\ 0 < x < L,\ with\ w(L) = 0$$

- In the first integral, we have product of 1^{st} order derivatives of *u* and *w*. We need to have functions for which this integral exists.
- Thinking for a moment (because of the symmetry in *u* and w) that *u*=*w*, we want trial *u* and weight *w* functions that are 1^{st} derivative square integrable, i.e. functions $v(x) \in H^1(I)$, $I = [0, L]$ such that:

$$\int_0^L EA\left(\frac{dv}{dx}\right)^2 dx < \infty, EA > 0$$

Continuous functions with piecewise continuous derivatives

- Let us select as a function space to define the weak form the following:

$$V = \{ v : v \; is \; continuous \; on \; I = [0, L], v' \; is \; piecewise \; continuous \; and \; bounded \; on \; I \}$$

- Can you verify that this type of functions will work for our weak form?
- A function in *V* is piecewise continuously differentiable, i.e. its first derivative is continuous except at selected points.
- Find $u \in V, with \; u(L) = \bar{u},$

  such that $\forall w \in V, with \; w(L) = 0,$

  the following holds:

$$\int_0^L EA \frac{du}{dx}\frac{dw}{dx} dx = wA\bar{t} \,|_{x=0} + \int_0^L bw dx$$

- A function with *V*-continuity, can have discontinuity (kinks) in its slope.

C^{-1} continuity

- Consider functions (C^{-1} continuity) with discontinuities (jumps) and discontinuities in slope (kinks)
- Such functions that are not infinite have 1st order derivatives that are integrable (you can integrate delta functions!)
- However, note that in our weak form we have a product of the 2 derivatives: $\int_0^L EA \frac{du}{dx}\frac{dw}{dx}dx$
- If both *u* and *w* are in C^{-1} and have a discontinuity at the same point *a* of magnitudes α and β, respectively, then the 1st integral in the eq. above takes the form:

$$\int_0^L \alpha\beta\delta^2(x-a)dx \longleftarrow$$

This integral does not exist!

- So the C^{-1} functions are not good candidates for our weight and trial functions.

C^0 functions

- Let us define the C^0-continuity functions as follows:

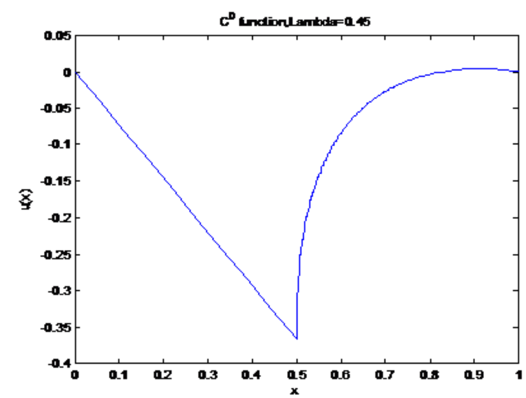
$$C^0(\bar{I}) = \left\{ v : v \text{ is a continuous function defined on } \bar{I} = [0, L] \right\}$$

- These functions in general are not appropriate as weight and trial functions in our weak form.
- Indeed, consider the following function:

$$u(x) = \begin{cases} -(\frac{1}{2})^{\lambda} x & x \le \frac{1}{2} \\ (x-\frac{1}{2})^{\lambda} - (\frac{1}{2})^{\lambda} x & x > \frac{1}{2} \end{cases}$$

- It is C^0 continuous for all $\lambda > 0$, however an H^1 function (e.g. 1st derivative square integrable) only for $\lambda > 1/2$. Verify this by computing $\int_0^1 \left(\frac{du}{dx} \right)^2 dx$ analytically.

C^0 continuous but not H^1 function →



Continuous functions with piecewise continuous derivatives versus H^1

- We will see in an upcoming lecture, that in finite element analysis we consider piecewise polynomial (linear, quadratic, etc.) functions on subdivisions (triangulations) of a bounded domain $\bar{I}=[0,L]$ on elements.
- Our selection of finite element approximation space $V_h \subset H^1(I)$, is satisfied in this case by selecting $V_h \subset C^0(\bar{I})$
- This is the case since $v \in C^0(\bar{I})$ will be shown to be continuous between the boundaries of adjacent elements and that its first derivative exists and is piecewise continuous so that $v \in H^1(I)$

Weak formulation in H^1

- Find $u \in V, with\ u(L) = \bar{u},$

  such that $\forall w \in V, with\ w(L) = 0,$

  the following holds:

$$\int_0^L EA \frac{du}{dx}\frac{dw}{dx} dx = wA\bar{t}\,|_{x=0} + \int_0^L bw dx$$

$$V = \{v : v \text{ is continuous on } I = [0, L], v' \text{ is piecewise continuous and bounded on } I\}$$

- When producing weak (variational) formulations of boundary value problems, it is (from mathematical point of view) natural and very useful to work with function spaces that are slighlty larger (i.e. spaces that contain more functions) than the spaces of continuous functions with piecewise continuous derivatives introduced earlier.
- It is also useful to endow *V* with scalar products and corresponding norms related to the actual boundary value problem.

The weak form in H^1

- We modify the weak problem statement as follows:

Find $u \in H^1, \text{with } u(L) = \bar{u},$
such that $\forall w \in H^1, \text{with } w(L) = 0,$
the following holds:

$$\int_0^L EA \frac{du}{dx}\frac{dw}{dx} dx = wA\bar{t}\,|_{x=0} + \int_0^L bw dx$$

- **The space H^1, is the space of 1st derivative square integrable functions.**

$$\underbrace{W_{\text{int}}(u)}_{\text{Internal strain energy}} = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx < \infty, EA > 0$$

we will later introduce $\sqrt{W_{\text{int}}(u)}$ as the energy norm.

- H^1 is the largest space we can write the above weak form and it naturally arises from the specific BVP. We will also see that the basic error estimate for the FE is in the H^1 norm.

Square integrable functions $L_2(I)$

- Consider in 1D the interval *I=(0,L).* We define the space of square integrable functons on *I* as:

$$L_2(I)=\left\{v: v \text{ is defined on } I \text{ and } \int_I v^2 dx<\infty\right\}$$

- The space $L_2(I)$ is a Hilbert space with the (L_2-) scalar (inner) product: $(v,w)_{L_2(I)}=\int_I vwdx$

  and corresponding (L_2-) norm: $\|v\|_{L_2(I)}=\left(\int_I v^2 dx\right)^{1/2}=(v,v)^{1/2}$

- By Cauchy's inequality:

$$|(v,w)_{L_2(I)}|\leq\|v\|_{L_2(I)}\|w\|_{L_2(I)}$$

- For this class, think of $v\in L_2(I)$ as a piecewise continuous function, possibly unbounded, such that: $\int_I v^2 dx<\infty$.

- As an example, for $x\in I=(0,1), v(x)=x^{-\beta}\in L_2(I)$ if $\beta<1/2$.

First derivative square integrable functions $H^1(I)$

- We formally introduce the space $H^1(I)$, $I=(0,L)$ as follows:

$$H^1(I)=\left\{v: v \text{ and } v' \equiv \frac{dv}{dx} \text{ belong to } L_2(I)\right\}$$

- The space $H^1(I)$ consists of the functions defined on I which together with their 1st derivatives are square integrable.
- This space is equipped with the following inner (scalar) product:

$$(v,w)_{H^1(I)}=\int_I (vw+v'w')dx$$

- The corresponding norm is: $\|v\|_{H^1(I)}=\left(\int_I \left(v^2+(v')^2\right)dx\right)^{1/2}$

- When $v(0)=0$ and/or $v(L)=0$, in literature, we often denote the corresponding space as

$$H_0^1(I)=\left\{v \in H^1(I): v(L)=0\right\}$$

Recall smoothness conditions for beams

- Recall that beam shape functions were C^1 continuous – both displacements and slopes were continuous (4th other ODEs).

$$q = -\frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI \frac{d^2 u_y}{dx^2} \right)$$

$$U^e = \int_{\Omega^e} \frac{E^e I^e}{2} \kappa^2 dx = \int_{\Omega^e} \frac{E^e I^e}{2} \left(\frac{d^2 u_y^e}{dx^2}\right)^2 dx$$

Weak form→ Strong form

- We will here show that if *u* satisfies the weak form, it is also a solution of the strong form. Let *u*:

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = wA\bar{t}\,|_{x=0} + \int_0^L bwdx,\ \forall w(x)\ in\ 0<x<L,\ with\ w(L)=0$$

- Integrate backwards the first term as follows:

$$EA\frac{du}{dx}w\,|_0^L - \int_0^L \frac{d}{dx}\left(EA\frac{du}{dx}\right)wdx = wA\bar{t}\,|_{x=0} + \int_0^L bwdx,\ \forall w(x)\ in\ 0<x<L,\ with\ w(L)=0 \Rightarrow$$

$$\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b\right]wdx + wA(\bar{t}+\sigma)\,|_{x=0} = 0,\ \forall w(x)\ in\ 0<x<L,\ with\ w(L)=0$$

- Since *w(x)* is arbitrary, select $w = \psi(x)\left(\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b\right)$ with $\psi(x)>0$ a smooth function with $\psi(0)=\psi(L)=0$ e.g. $\psi = x(L-x)$

Weak form→ Strong form

$$\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx} \right) + b \right] w dx + wA(\bar{t} + \sigma)|_{x=0}, \forall w(x) \; in \; 0 < x < L, \; with \; w(L) = 0$$

- With this selection of *w*, the Eq. above becomes:

$$\int_0^L \psi(x) \left[\frac{d}{dx}\left(EA\frac{du}{dx} \right) + b \right]^2 dx = 0$$

- With $\psi(x) > 0$, the above can be true only if:

$$\frac{d}{dx}\left(EA\frac{du}{dx} \right) + b = 0, 0 < x < L$$

- This proves that if *u* is a solution of the weak form, it also satisfies the ODE strong form. It remains to show that it satisfies the natural BC.

Weak form→ Strong form

$$\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx} \right) + b \right] w dx + wA(\bar{t} + \sigma)\,|_{x=0} = 0,\ \forall w(x)\ in\ 0 < x < L,\ with\ w(L) = 0$$

- Returning to the above equation, we have:

$$wA(\bar{t} + \sigma)\,|_{x=0} = 0,\ \forall w(x)\ with\ w(L) = 0$$

- Since *w* is arbitrary, select it such that *w(0)=1, w(L)=0*, then

$$\sigma = -\bar{t}\ at\ x = 0$$

- In conclusion, if *u* is a solution of the weak form, it also satisfies the complete strong form of our BVP.

Strong form: Axial loading of an elastic bar

- Let us generalize the earlier examined problem by allowing arbitrary part of the boundary Γ_t with prescribed displacement and the remaining part Γ_u with prescribed traction:

$$\frac{d}{dx}(EA\frac{du}{dx})+b=0,\ 0<x<L$$

$$u=\bar{u}\ on\ \Gamma_u$$

$$\Gamma_u \cap \Gamma_t = 0$$

$$\Gamma_u \cup \Gamma_t = \Gamma$$

$$\sigma n = E\frac{du}{dx}n=\bar{t}\ on\ \Gamma_t$$

- The nornal vector *n* is introduced to allow arbitrary application of the traction BC (recall $\bar{t}>0$). If *n=1* (right boundary), then $\sigma=\bar{t}\ on\ \Gamma_t$. If a positive force is applied at the left boundary (*n=-1*), then $\sigma=-\bar{t}\ on\ \Gamma_t$.

Weak form: Axial loading of an elastic bar

- Multiplying the ODE with *w* and integrating by parts gives:

$$EA\frac{du}{dx}wn|_{\Gamma} - \int_{\Omega} EA\frac{du}{dx}\frac{dw}{dx}dx + \int_{\Omega} bw dx = 0, \forall w(x)\ in\ 0<x<L, with\ w=0\ on\ \Gamma_u \Rightarrow$$

$$\int_{\Omega} EA\frac{du}{dx}\frac{dw}{dx}dx = Aw\bar{t}\,|_{\Gamma_t} + \int_{\Omega} bw dx, \forall w(x)\ in\ 0<x<L, with\ w=0\ on\ \Gamma_u$$

- We require that this holds for $u(x) \in H^1$, where $u = \bar{u}\ on\ \Gamma_u$ and for all $w(x) \in H^1$ with $w = 0\ on\ \Gamma_u$. H^1 is the space of 1st derivative square integrable functions.
- As before, the weak form is symmetric in *u* and *w* (first derivatives for both *u* and *w*). This will lead to a symmetric stiffness matrix.

Summary of weak forms for the two problems

- Axial deformation problem: Find $u(x) \in U$, $U = \{u(x) \in H^1, u = \bar{u} \; in \; \Gamma_u\}$

$$\int_\Omega EA \frac{du}{dx}\frac{dw}{dx} dx = Aw\bar{t} \,|_{\Gamma_t} + \int_\Omega bw dx, \; \forall w(x) \in U_0,$$

$$U_0 = \{w(x) \in H^1, w = 0 \; in \; \Gamma_u\}$$

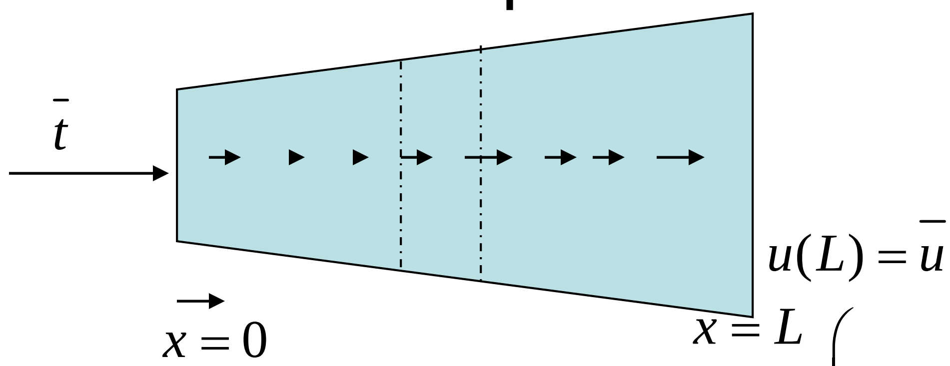
- Heat conduction problem: Find $u(x) \in U$, $U = \{u(x) \in H^1, u = \bar{u} \; in \; \Gamma_u\}$

$$\int_\Omega \frac{du}{dx} Ak \frac{dw}{dx} dx = -Aw\bar{q} \,|_{\Gamma_q} + \int_\Omega bw dx, \; \forall w(x) \in U_0,$$

$$U_0 = \{w(x) \in H^1, w = 0 \; in \; \Gamma_u\}$$

Minimum potential energy

- We used earlier the principle of mimimum potential energy to derive the FEM eqs.
- How is this method related to the weak form? Let us revisit the elastic bar problem.



$$\text{Find} \quad \underbrace{u(x) \in U,}_{\substack{\text{Kinematically admissible displacements,} \\ \text{satisfy kinematic (displacement) boundary} \\ \text{conditions } u(L)=\bar{u}}} \qquad \text{such that} \; \underset{u(x)}{\underbrace{\min}} \underbrace{\frac{1}{2}\int_{\Omega} AE\left(\frac{du}{dx}\right)^2 dx}_{W_{\text{int}}} - \left(\underbrace{\int_{\Omega} u b\, dx + (uA\bar{t})\big|_{\Gamma_t}}_{W_{ext}} \right)$$

Minimum potential energy

$$Find\ u(x) \in U,\ such\ that\ \min_{u(x)} \mathrm{P},\ where$$

$$P = \underbrace{\frac{1}{2}\int_{\Omega} AE(\frac{du}{dx})^2 dx}_{W_{\text{int}}} - \left(\underbrace{\int_{\Omega} ubdx + (uA\bar{t})|_{\Gamma_t}}_{W_{ext}} \right)$$

- We will show that the minimizer of the potential energy satisfies the weak form (and thus the strong form).
- *P(u(x))* is a functional – takes a function as an input and returns a scalar! We need to use **calculus of variations** for its mimimization.
- A ***variation of the function u*** is denoted by $\delta u(x) \equiv \zeta w(x)$ where $w(x)$ is an arbitrary function, and $0<\zeta<1$ is a very small positive number.

Variation of a functional

- The corresponding change in the functional is called the variation in the functional and denoted by δW , which is defined by

$$\delta P = \mathrm{P(u(x)+}\zeta\,\mathrm{w(x))\text{-}P(u(x))=P(u(x)+}\delta\mathrm{u(x))\text{-}P(u(x))}$$

- From the principle of minimum potential energy, it is clear that the function $u(x) + \zeta\, w(x)$ must still be in U. To meet this condition, $w(x)$ must be smooth and vanish on the essential boundaries, i.e. $w \in U_0$
- So *w(x)* vanishes at the parts of the boundary with essential boundary conditions!

Variation of the internal energy

- Let us evaluate the variation of the first term in W_{int}.

$$\delta W_{\text{int}} = \frac{1}{2}\int_{\Omega} AE\left(\frac{du}{dx}+\varsigma\frac{dw}{dx}\right)^2 dx - \frac{1}{2}\int_{\Omega} AE\left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{1}{2}\int_{\Omega} AE\left(\left(\frac{du}{dx}\right)^2 + 2\varsigma\frac{du}{dx}\frac{dw}{dx}+\varsigma^2\left(\frac{dw}{dx}\right)^2\right)dx - \frac{1}{2}\int_{\Omega} AE\left(\frac{du}{dx}\right)^2 dx$$

- Dropping the higher order terms in ζ, leads to:

$$\delta W_{\text{int}} = \varsigma\int_{\Omega} AE\frac{du}{dx}\frac{dw}{dx}dx$$

Variation of the external work

- Let us evaluate the variation of the 2nd term, W_{ext}.

$$\delta W_{ext} = \int_{\Omega} (u+\varsigma w) b dx + (u+\varsigma w) A\bar{t}\,|_{\Gamma_t} - \int_{\Omega} u b dx - uA\bar{t}\,|_{\Gamma_t} = \int_{\Omega} \varsigma w b dx + \varsigma w\bar{t}\,|_{\Gamma_t}$$

- Combining the last 2 terms, the variation of the potential energy is given as:

$$\delta P = \varsigma \int_{\Omega} AE \frac{du}{dx}\frac{dw}{dx} dx - \int_{\Omega} \varsigma w b dx - \varsigma w A\bar{t}\,|_{\Gamma_t}$$

- Minimization of the potential energy requires: $\delta P = 0$, for any $\delta u = \varsigma w$, *i.e. for any* ς:

$$\int_{\Omega} AE \frac{du}{dx}\frac{dw}{dx} dx = \int_{\Omega} w b dx + wA\bar{t}\,|_{\Gamma_t}$$

Variational principle

$$\int_{\Omega} AE \frac{du}{dx}\frac{dw}{dx}dx = \int_{\Omega} wbdx + wA\bar{t}\,|_{\Gamma_t}$$

- This is identical with our weak form. So minimization of the potential energy leads to the same solution as the weak problem, thus

$$Strong\ problem \Leftrightarrow Weak\ problem \Leftrightarrow Variational\ problem$$

- Note that now we understand why in the weak form we considered *w* to vanish on the boundary with essential conditions: $\delta u = \varsigma w(x)$ and the miminization is wrt functions *u(x)* that satisfy the essential BCs, thus

$$\delta u\,|_{\Gamma_u} = \varsigma w(x)\,|_{\Gamma_u} = 0 \Rightarrow w(x)\,|_{\Gamma_u} = 0$$

Principle of virtual work

- The minimization of *P*, $\delta P = 0,$

$$\delta P = \varsigma \int_{\Omega} AE \frac{du}{dx}\frac{dw}{dx} dx - \int_{\Omega} \varsigma w b dx - \varsigma w A\bar{t}\,|_{\Gamma_t} = 0$$

- This can also be written as $(\delta u \equiv \zeta w)$:

$$\delta P = \int_{\Omega} \underbrace{E \frac{du}{dx}}_{\sigma} \underbrace{\frac{d\delta u}{dx}}_{\delta\varepsilon} \underbrace{A dx}_{dV} - \int_{\Omega} \delta u b dx - \delta u A\bar{t}\,|_{\Gamma_t} = 0$$

- This is simplified as:

$$\delta P = \underbrace{\int_{\Omega} \sigma \delta\varepsilon dV}_{\delta W_{\text{int}}} - \left(\underbrace{\int_{\Omega} \delta u b dx + \delta u A\bar{t}\,|_{\Gamma_t}}_{\delta W_{ext}} \right) = 0$$

Principle of virtual work

$$\delta P = \underbrace{\int_{\Omega} \sigma \delta \varepsilon dV}_{\delta W_{\text{int}}} - \left(\underbrace{\int_{\Omega} \delta u b dx + \delta u A \bar{t} \,|_{\Gamma_t}}_{\delta W_{ext}} \right) = 0$$

- The internal work done by the actual stresses on the arbitrary strains $\delta\varepsilon \equiv \frac{d\delta u}{dx}$ induced by the arbitrary displacements $\delta u \in U_0$ is equal to the external work done by the body forces b and applied tractions $\bar{t}$ on the arbitrary displacement field δu .

- This statement is of course the same as the weak form but presented in ` mechanics' language.

Limitations of the variational principle

- There are many problems to which the variational principle is not applicable. For example, variational principles cannot be developed for the advection-diffusion equation.

- Variational principles can only be developed for systems that are self-adjoint.

- The weak form for the advection-diffusion equation is not symmetric, and it is not a self-adjoint system.

Example problem 1

- Derive the weak form for the following BVP:

$$\frac{d}{dx}(10\frac{du}{dx})+2x=0, 1<x<3, \frac{du}{dx}(1)=0.1, u(3)=0.001$$

- We multiply by *w* and integrate from *1* to *3*, for every *w, w(3)=0*.

$$\int_1^3\left(\frac{d}{dx}(10\frac{du}{dx})+2x\right)wdx=0, \forall w, w(3)=0$$

- Integration by parts of the 1st term gives:

$$10\frac{du}{dx}w\Big|_{x=1}^{x=3}-\int_1^3 10\frac{du}{dx}\frac{dw}{dx}dx+\int_1^3 2xwdx=0, \forall w, w(3)=0\Rightarrow$$

$$\cancel{10\frac{du}{dx}(3)\underbrace{w(3)}_{0}}-10\underbrace{\frac{du}{dx}(1)}_{0.1}w(1)-\int_1^3 10\frac{du}{dx}\frac{dw}{dx}dx+\int_1^3 2xwdx=0, \forall w, w(3)=0\Rightarrow$$

$$10\int_1^3\frac{du}{dx}\frac{dw}{dx}dx=-w(1)+2\int_1^3 xwdx, \forall w, w(3)=0$$

Example problem 2

$$\int_1^3 10\frac{du}{dx}\frac{dw}{dx}dx = -w(1) + 2\int_1^3 xwdx, \ \forall w, w(3) = 0$$

- For the problem discussed earlier, consider a trial (candidate) solution of the form $u(x) = \alpha_0 + \alpha_1(x-3)$ and a weight function of the same form, $w(x) = \beta_0 + \beta_1(x-3)$. Obtain a solution to the weak form shown above.
  - Check if the equilibrium equation in the strong form is satisfied?
  - Check if the natural boundary condition is satisfied?

- Since *w(3)=0*, $w(x) = \beta_0 + \beta_1(x-3) \Rightarrow w(x) = \beta_1(x-3) \ and \ \frac{dw}{dx} = \beta_1$

Example problem 2

$$\int_1^3 10\frac{du}{dx}\frac{dw}{dx}dx = -w(1) + 2\int_1^3 xwdx, \forall w, w(3) = 0$$

$$u(x) = \alpha_0 + \alpha_1(x-3), \frac{du}{dx} = \alpha_1, w(x) = \beta_1(x-3), \frac{dw}{dx} = \beta_1$$

- Substitution in the weak form gives:

$$\int_1^3 10\alpha_1\beta_1 dx = 2\beta_1 + 2\int_1^3 x\beta_1(x-3)dx, \forall \beta_1 \Rightarrow \qquad 20\alpha_1\beta_1 = 2\beta_1 - \frac{20}{3}\beta_1, \forall \beta_1 \Rightarrow$$

$$20\alpha_1 = 2 - \frac{20}{3} \Rightarrow \alpha_1 = -\frac{7}{30}$$

- So the solution is: $u(x) = 0.001 - \frac{7}{30}(x-3)$ $(\textit{here we used } u(3) = 0.001)$
- Is the strong form satisfied?

$$\frac{d}{dx}(10\frac{du}{dx}) + 2x = -\frac{d}{dx}(10\frac{7}{30}) + 2x = 2x \neq 0, \textit{ not satisfied}$$

- Is the natural BC satisfied? $\frac{du}{dx}(1) = -\frac{7}{30} \neq 0.1, \textit{ not satisfied}$

Example problem 3

- Find the weak form of the following BVP:

$$k\frac{d^2u}{dx^2} - \lambda u + 2x^2 = 0,\ 0 < x < 1,\ u(0) = 1,\ u(1) = -2,\ k, \lambda \ constants$$

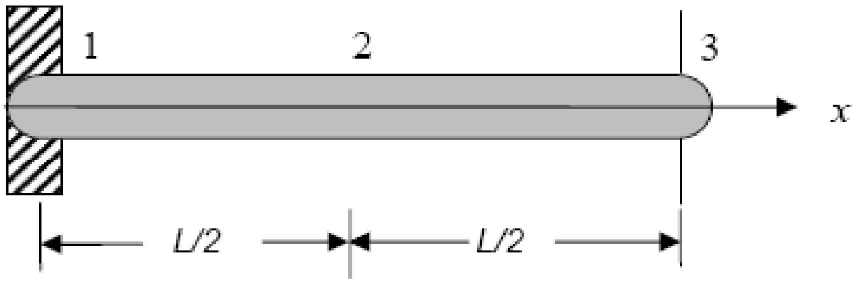
- Multiplying by *w* with *w(0)=w(1)=0* and integration from *0* to *1* gives:

$$\int_0^1 \left(k\frac{d^2u}{dx^2} - \lambda u + 2x^2 \right) w dx = 0,\ \forall w, w(0) = w(1) = 0 \Rightarrow$$

$$\cancel{k\frac{du}{dx}(1)w(1)} - \cancel{k\frac{du}{dx}(0)w(0)} - k\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx - \lambda\int_0^1 uwdx + 2\int_0^1 x^2 wdx = 0,\ \forall w, w(0) = w(1) = 0 \Rightarrow$$

$$k\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \lambda\int_0^1 uwdx = 2\int_0^1 x^2 wdx,\ \forall w, w(0) = w(1) = 0$$

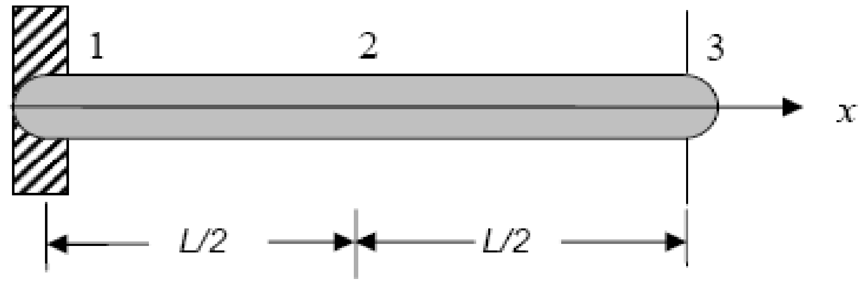
Example problem 4



- Consider a bar subjected to linear body force $b(x)= Cx$. The bar has a constant cross-sectional area A and Young's modulus E. Assume quadratic trial solution and weight functions:
  $u(x)=\alpha_1+\alpha_2 x+\alpha_3 x^2$
  $w(x)= \beta_1+\beta_2 x+\beta_3 x^2$
- For what value of α_i is $u(x)$ kinematically admissible?
- Using the weak form, set up the equations for α_i and solve.
- Solve the problem using two 2-node elements of equal size like the elements we used in the lecture on truss structures (direct method). Approximate the external load at node 2 by integrating the body force from $x=L/4$ to $x=3L/4$; Likewise compute the external at node 3 by integrating the body force from $x=3L/4$ to $x=L$.

Example problem 4

- On $\Gamma_u : u(0) = 0 \Rightarrow \alpha_1 = 0$
- The constraint on the weight functions is then: $w(0) = 0 \Rightarrow \beta_1 = 0$



$$u(x) = \alpha_2 x + \alpha_3 x^2, \frac{du}{dx} = \alpha_2 + 2\alpha_3 x$$

$$w(x) = \beta_2 x + \beta_3 x^2, \frac{dw}{dx} = \beta_2 + 2\beta_3 x$$

- The weak for this problem is the following:

$$\int_0^L AE \frac{du}{dx}\frac{dw}{dx} dx - \int_0^L wb dx - w A \bar{t} \big|_{\Gamma_t} = 0 \Rightarrow$$

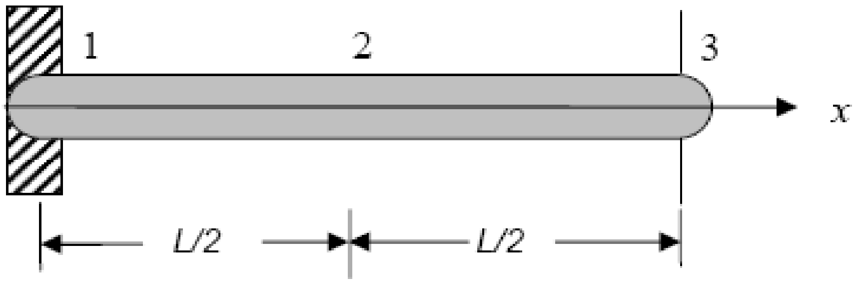
$$\int_0^L AE\left(\alpha_2 + 2\alpha_3 x\right)\left(\beta_2 + 2\beta_3 x\right) dx = C \int_0^L \left(\beta_2 x + \beta_3 x^2\right) x dx \Rightarrow$$

$$\frac{AE}{C}\int_0^L \left(\beta_2\alpha_2 + 2\beta_2\alpha_3 x + 2\beta_3\alpha_2 x + 4\beta_3\alpha_3 x^2\right) dx = \int_0^L \left(\beta_2 x^2 + \beta_3 x^3\right) dx \Rightarrow$$

Example problem 4

$$\frac{AE}{C}\int_0^L \left(\beta_2\alpha_2 + 2\beta_2\alpha_3 x + 2\beta_3\alpha_2 x + 4\beta_3\alpha_3 x^2\right)dx$$

$$= \int_0^L \left(\beta_2 x^2 + \beta_3 x^3\right)dx \;\forall \beta_2, \beta_3 \Rightarrow$$



$$\frac{AE}{C}\int_0^L \left(\beta_2\alpha_2 + (2\beta_2\alpha_3 + 2\beta_3\alpha_2)x + 4\beta_3\alpha_3 x^2\right)dx = \int_0^L \left(\beta_2 x^2 + \beta_3 x^3\right)dx \;\forall \beta_2, \beta_3 \Rightarrow$$

$$\frac{AE}{C}\left(\beta_2\alpha_2 L + (2\beta_2\alpha_3 + 2\beta_3\alpha_2)\frac{L^2}{2} + 4\beta_3\alpha_3\frac{L^3}{3}\right) = \beta_2\frac{L^3}{3} + \beta_3\frac{L^4}{4} \;\forall \beta_2, \beta_3 \Rightarrow$$

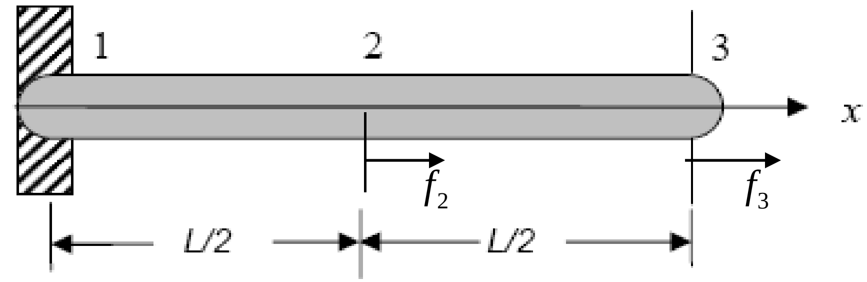
$$\beta_2\left\{\frac{AE}{C}\left(\alpha_2 L + \alpha_3 L^2\right) - \frac{L^3}{3}\right\} + \beta_3\left\{\frac{AE}{C}\left(\alpha_2 L^2 + 4\alpha_3\frac{L^3}{3}\right) - \frac{L^4}{4}\right\} = 0 \;\forall \beta_2, \beta_3 \Rightarrow$$

$$\begin{bmatrix} \frac{AEL}{C} & \frac{AEL^2}{C} \\ \frac{AEL^2}{C} & \frac{4}{3}\frac{AEL^3}{C} \end{bmatrix}\begin{Bmatrix}\alpha_2 \\ \alpha_3\end{Bmatrix} = \begin{Bmatrix}\frac{L^3}{3} \\ \frac{L^4}{4}\end{Bmatrix} \Rightarrow \begin{Bmatrix}\alpha_2 \\ \alpha_3\end{Bmatrix} = \begin{Bmatrix}\frac{7CL^2}{12AE} \\ \frac{-CL}{4AE}\end{Bmatrix} \Rightarrow \quad u(x) = \frac{7CL^2}{12AE}x - \frac{CL}{4AE}x^2$$

Example problem 4

$$f_2 = \int_{L/4}^{3L/4} Cx\,dx = \frac{CL^2}{4}$$

$$f_3 = \int_{3L/4}^{L} Cx\,dx = \frac{7CL^2}{32}$$



$$K^{(1)} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, K^{(2)} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, K = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Our final system of assembled equations is the following:

$$\frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ \frac{CL^2}{4} \\ \frac{7CL^2}{32} \end{Bmatrix} \Rightarrow \frac{2AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \frac{CL^2}{4} \\ \frac{7CL^2}{32} \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \frac{15CL^3}{64AE} \\ \frac{11CL^3}{32AE} \end{Bmatrix}$$

$$r_1 = \frac{2AE}{L}(-u_2) = \frac{2AE}{L}\left(-\frac{15CL^3}{64AE}\right) = -\frac{15}{32}CL^2$$

Discretized weak form with finite elements

- Consider the following BVP:

$$-u''(x) = f(x),\ 0 < x < 1$$
$$u(0) = u(1) = 0$$

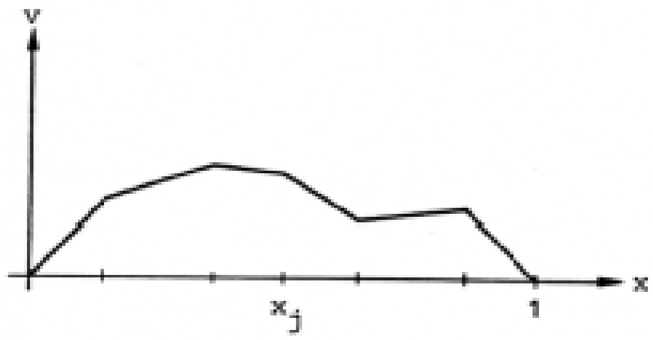
- We multiply the equation $-u'' = f$ by an arbitrary function $w \in V$
- Integrate over (0, 1) which gives:

  $-(u'', w) = (f, w)$ [We use the following notation: $(f,g) \equiv \int_0^1 fg\,dx$]
- We now integrate the left-hand side by parts using the fact that $w(0) = w(1) = 0$ to get:

  $-(u'', w) = -u'(1)w(1) + u'(0)w(0) + (u', w') = (u', w')$
- We conclude that: $(u', w') = (f, w), \forall w \in V$ *or equivalently* : $\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx = \int_0^1 fw\,dx\ \forall w \in V$
- *We are next going to look for approximate solutions using piece-wise linear C^0 (continuous) functions.*

Finite element discretization: Piecewise linear functions

- Construct a finite-dimensional subspace $V_h \subset V$ as follows:
- Define intervals $I_j = (x_{j-1}, x_j)$ of length $h_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, (M + 1)$ and set $h = \max_j h_j$.
- Let V_h be the set of functions v such that:
  - v is linear on each subinterval I_j
  - v is continuous on $[0, 1]$ and
  - $v(0) = v(1) = 0$.
- The quantity $h = \max_j h_j$ is a measure of how fine the partition is.



Piecewise linear basis functions $N_j(x)$, $j = 1, 2, \ldots, M$

- A function $v \in V^h$ has the representation:

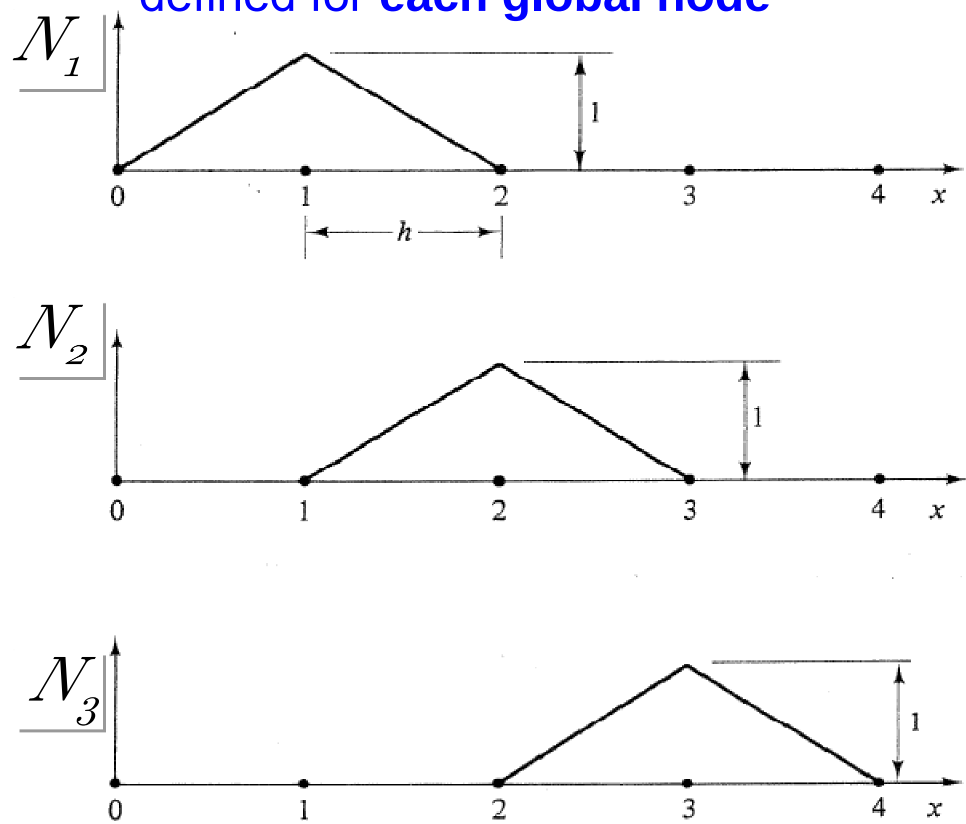
$$v(x) = \sum_{i=1}^{M} \underbrace{v(x_i)}_{\substack{\text{Nodal values} \\ \text{of } v(x)}} N_i(x), \quad 0 \le x \le 1$$

where for i,j=1,…,M

$$N_j(x_i) = \delta_{ij} \equiv \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

- The space V_h is a linear space of dimension M with basis $\{N_j\}$, $i=1,..,M$

These are **global basis functions** each one defined for **each global node**



$$N_i(x) \in H_0^1(0,1)$$

This is not much different from a Fourier expansion – the difference is that we use piecewise linear functions instead of sin and cos functions

Linear system of equations: Stiffness and load

- We approximate the solution as $u_h(x) = \sum_{j=1}^{M} u_h(x_j) N_j(x),\ 0 \le x \le 1$ with $u_h(x_j)$ the nodal values of $u_h(x)$.
- Recall our weak formulation: $(u', w') = (f, w), \forall w \in V$
- It now takes the form:

$$Find\ u_h : (u_h^{'}, w_h^{'}) = (f, w_h),\ \forall w_h \in V^h$$

- The space V_h is spanned by M functions N_i, thus the weak form takes the form:

$$Find\ u_h : (u_h^{'}, N_i^{'}) = (f, N_i),\ i = 1, 2, \ldots, M$$

- We now left with substituting in this equation: $u_h(x) = \sum_{j=1}^{M} u_h(x_j) N_j(x).$

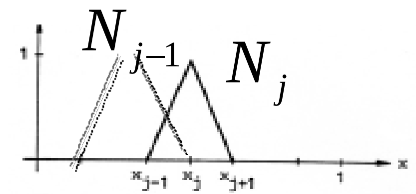
$$Find\ u_h(x_j) : \sum_{j=1}^{M} \underbrace{(N_j^{'}, N_i^{'})}_{K_{ij}} \underbrace{u_h(x_j)}_{d_j} = \underbrace{(f, N_i)}_{F_i},\ i = 1, 2, \ldots, M$$

Properties of the stiffness matrix

$$Find\ u_h : \sum_{j=1}^{M} \underbrace{(N_j', N_i')}_{K_{ij}} \underbrace{u_h(x_j)}_{d_j} = \underbrace{(f, N_i)}_{F_i},\ i = 1, 2, \ldots, M \quad where: K_{ij} = \int_0^1 N_i' N_j' dx,\ F_i = \int_0^1 f(x) N_i\, dx$$

- *K* is symmetric: $K_{ij} = K_{ji}$, $\quad (N_j', N_i') = (N_i', N_j'),\ i, j = 1, 2, \ldots, M$
- *K* is sparse (i.e. only a few elements of *K* are nonzero)

$$(N_j', N_j') = \int_{x_{j-1}}^{x_j} \frac{1}{h_j^2} dx + \int_{x_j}^{x_{j+1}} \frac{1}{h_{j+1}^2} dx = \frac{1}{h_j} + \frac{1}{h_{j+1}},\ j = 1, 2, \ldots, M$$



$$(N_j', N_{j-1}') = (N_{j-1}', N_j') = -\int_{x_{j-1}}^{x_j} \frac{1}{h_j^2} dx = -\frac{1}{h_j},\ j = 2, \ldots, M$$

$$(N_i', N_j') = 0, |i - j| > 1$$

- *K* is positive definite: Indeed for $\forall \eta \in \mathbb{R}^M$ we obtain:

$$\eta.K\eta = \sum_{i=1}^{M}\sum_{j=1}^{M} \eta_i K_{ij} \eta_j = \sum_{i=1}^{M}\sum_{j=1}^{M} \eta_i (N_i', N_j') \eta_j = (\underbrace{\sum_{i=1}^{M} \eta_i N_i'}_{\eta}, \sum_{j=1}^{M} \eta_j N_j') = (\eta, \eta) \geq 0$$

$$\eta.K\eta = 0,\ only\ \ for\ \eta = 0.$$

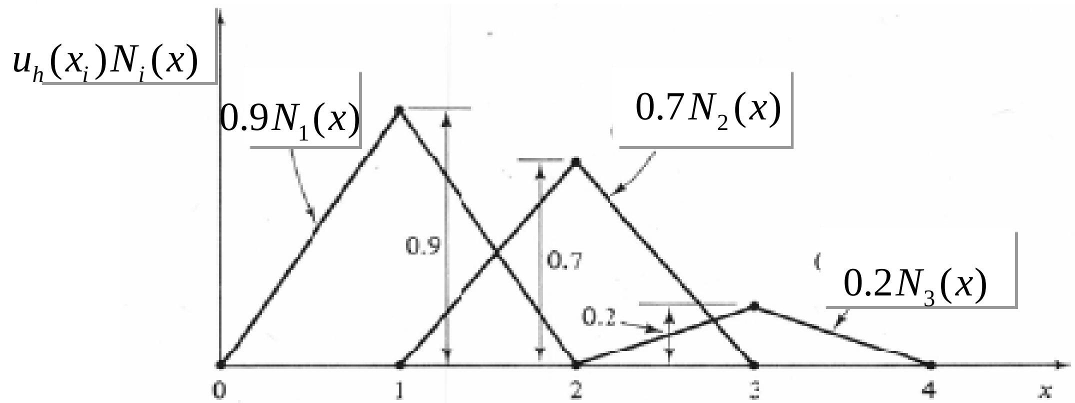
Properties of the stiffness matrix

- Since K is a positive definite matrix, we conclude that K is non-singular (note that the boundary conditions are already considered).
- It follows that the system $Kd = F$ has a unique solution.
- For the particular case of $h_j = h = 1/(M+1)$, the system $Kd = F$ becomes:

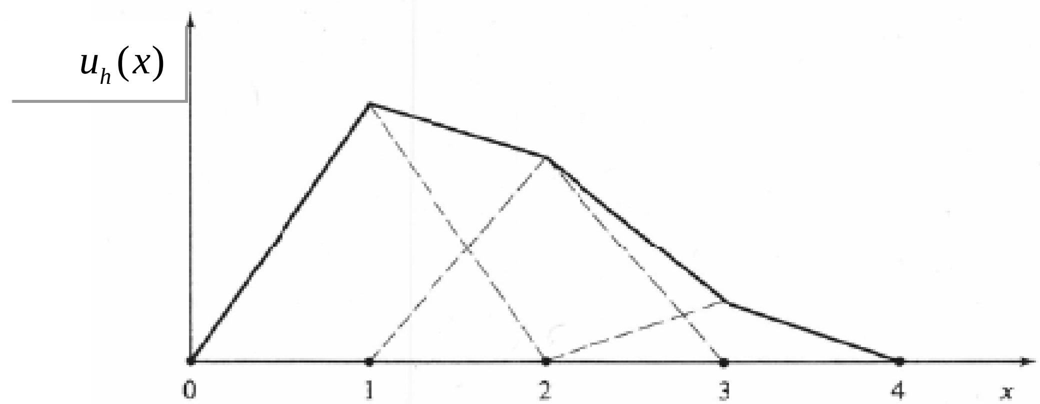
$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & . & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & 0 \\ . & 0 & -1 & 2 & -1 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ & & & & & -1 & 2 & -1 \\ & & & & & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ . \\ d_{M-1} \\ d_M \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ . \\ . \\ . \\ . \\ F_{M-1} \\ F_M \end{Bmatrix}$$

Final solution

- Let us assume that the solution obtained with 3 nodes (excluding the boundary nodes with *u=0*) is the following:



$$u_h(x) = u_h(x_1)N_1(x) + u_h(x_2)N_2(x) + u_h(x_2)N_3(x) =$$
$$0.9N_1(x) + 0.7N_2(x) + 0.2N_3(x)$$



- The piece-wise linear representation of the FE solution is shown here.