MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

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Mathematical preliminaries

- Let us consider a domain $\overline{\Omega}$ with interior Ω and boundary $\partial\Omega$.
- Usually the boundary $\partial\Omega$ is defined parametrically as follows: $x(s), y(s)$, where s is the arc length along $\partial\Omega$.
- For a function *f(x,y)* on the boundary $\partial\Omega$, we can write: $f(s) = f(x(s), y(s)), s \in \partial\Omega$.
- We denote the primary (scalar) variable in 2D as *u(x,y).* Until further notice, we assume that all functions used here are smooth enough for the operations shown to be valid.

Mathematical preliminaries

- Recall that the gradient $\vec{\nabla} u(x,y)$ is the rate of change of *u* as we move from *(x,y)* to nearby locations:

$$\vec{\nabla} u(x,y) = \frac{\partial u(x,y)}{\partial x}\vec{i} + \frac{\partial u(x,y)}{\partial y}\vec{j}$$

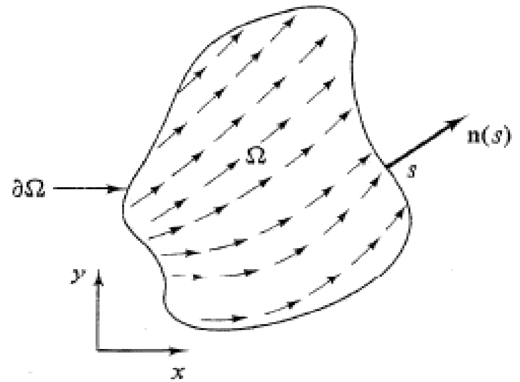
- Many times we will interpret this as a mathematical operation of the differential operator $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$ on $u(x,y)$.

- The gradient $\vec{\nabla} u(x,y)$ determines the rate of change of *u* at *(x,y)* in any direction
- The rate of change of *u* in a given direction $\vec{t} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ is given as:

$$\frac{du(x,y)}{dt} = \vec{\nabla} u(x,y) \bullet \vec{t} = \frac{\partial u(x,y)}{\partial x}\cos\theta + \frac{\partial u(x,y)}{\partial y}\sin\theta$$

Flux $\vec{\sigma}(x, y)$

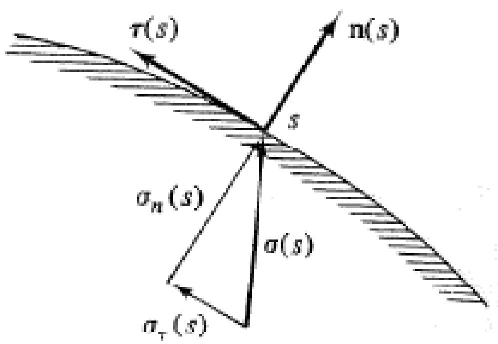
- Similarly to the gradient $\vec{\nabla} u(x, y)$ the flux $\vec{\sigma}(x, y)$ is a vector-valued function or vector field.



- The flux $\vec{\sigma}(s)$ at the boundary can be decomposed in normal and tangential components:

$$\sigma_n(s) = \vec{\sigma}(s) \bullet \vec{n}(s)$$

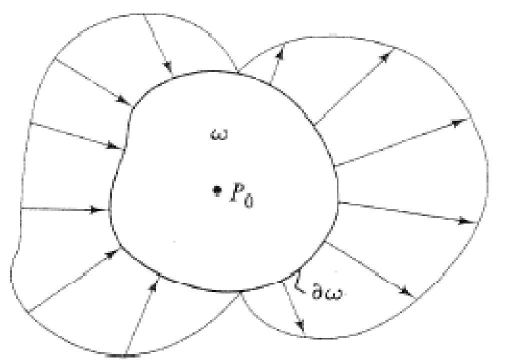
$$\sigma_\tau(s) = \vec{\sigma}(s) \bullet \vec{\tau}(s)$$



Flux $\vec{\sigma}(x, y)$

- Consider an arbitrary part $\omega \subset \Omega$ containing point (x_0, y_0) of the domain Ω. Consider the normal flux $\sigma_n(s) = \vec{\sigma}(s) \bullet \vec{n}(s)$ across the boundary $\partial\omega$. The total flux across this boundary is

$$\Sigma_\omega = \int_{\partial\omega} \sigma_n(s) ds$$

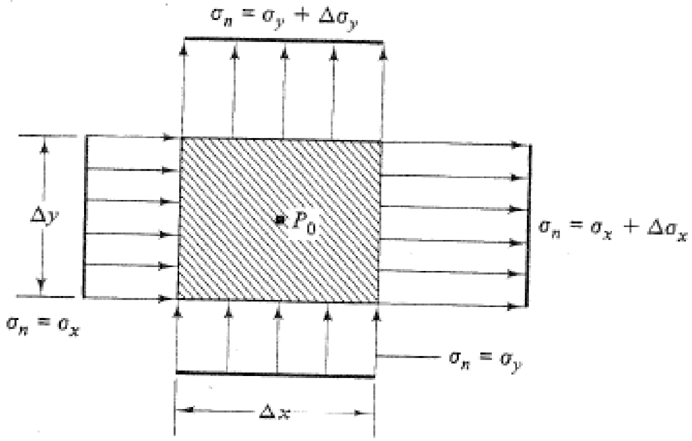


- If you divide Σ_ω by the area A_ω of the subregion, we can view the result as the amount of $\vec{\sigma}$ flowing into ω per unit area.

Divergence of the flux $\vec{\sigma}(x,y)$

- The limit of $\Sigma_\omega / A_\omega = \int_{\partial\omega} \sigma_n(s)ds / A_\omega$ as ω decreases in size always containing point (x_0, y_0) is called the divergence of the flux at P_0 :

$$div\vec{\sigma}(x_0, y_0)$$



- Taking as ω the square region shown, then:

$$\Sigma_\omega = \Delta\sigma_x \Delta y + \Delta\sigma_y \Delta x$$

- Dividing by the area $\Delta x \Delta y$ and taking the limit $\Delta x, \Delta y \to 0$:

$$div\vec{\sigma}(x,y) = \frac{\partial\sigma_x(x,y)}{\partial x} + \frac{\partial\sigma_y(x,y)}{\partial y} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right) \bullet \left(\sigma_x \vec{i} + \sigma_y \vec{j}\right) \equiv \vec{\nabla} \bullet \vec{\sigma}$$

So $\vec{\nabla} \bullet \vec{\sigma}$ is the density of the net flux per unit area at a point

Gauss divergence theorem

- The total flux out of the region Ω is then:

$$\Sigma = \int_{\Omega} \vec{\nabla} \bullet \vec{\sigma} dxdy$$

- We can now write using an earlier expression:

$$\Sigma = \int_{\Omega} \vec{\nabla} \bullet \vec{\sigma} dxdy = \int_{\partial\Omega} \vec{\sigma} \bullet \vec{n} ds$$

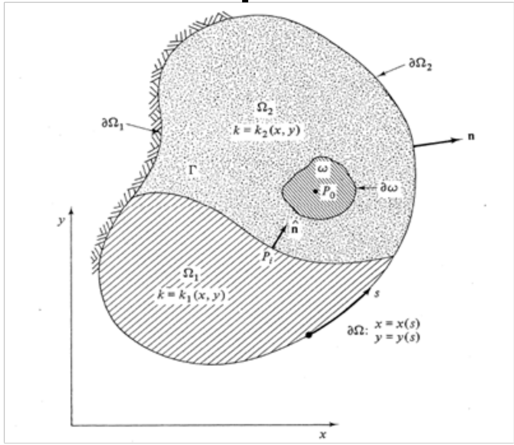
- This is the Gauss divergence theorem.

Constitutive equation and balance law

- Constitutive equation: We consider problems with the following constitutive equation between the flux $\vec{\sigma}$ and the state variable *u(x,y).*

$$\vec{\sigma}(x,y) = -\underbrace{k(x,y)}_{\substack{material\ \text{modulus} \\ k(x,y)>0\ in\ \overline{\Omega}}} \vec{\nabla}u(x,y)$$

- Conservation principle: Within any portion of the domain, the net flux across the boundary of that part must be equal to the total quantity produced by internal sources.



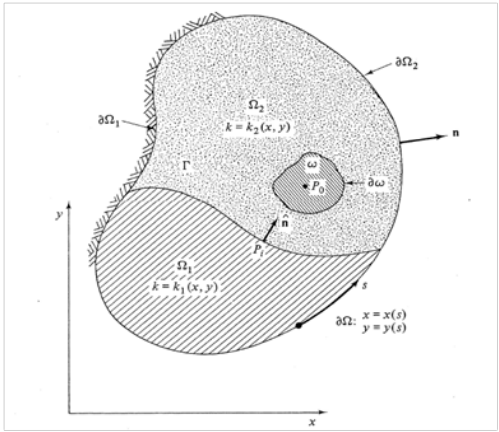
$$\int_{\partial\omega} \vec{\sigma} \bullet \vec{n} ds = \int_{\omega} f dx dy$$

f is source per unit area

Balance law

- Using the divergence theorem:

$$\int_{\partial\omega} \vec{\sigma} \bullet \vec{n} ds = \int_{\omega} f dx dy \Rightarrow \int_{\omega} (\vec{\nabla} \bullet \vec{\sigma} - f) dx dy = 0 \ \ for\ all\ \omega \subset \Omega.$$



- Since ω is arbitrary, we conclude that the local form of the balance equation is:

$$\vec{\nabla} \bullet \vec{\sigma}(x, y) = f(x, y)$$

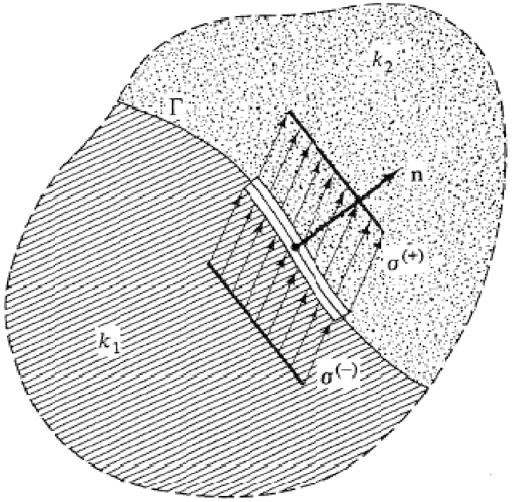
- To make things more interesting, we assume an additional source term proportional to *u(x,y):*

$$\vec{\nabla} \bullet \vec{\sigma}(x, y) + b(x, y) u(x, y) = f(x, y)$$

Conservation principle at interfaces

- Consider an interface Γ separating two materials with different modulus k_1 and k_2. Take a thin strip of the body around a point on this interface. The balance law takes the form:

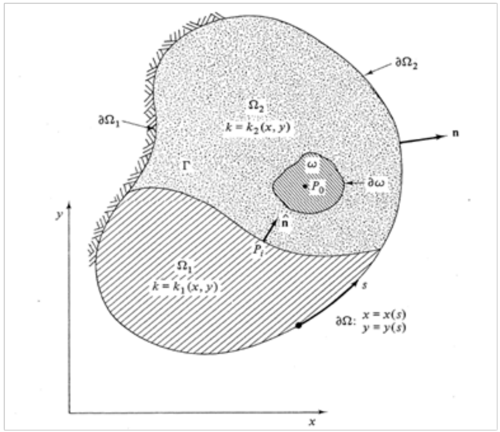
$$\Sigma = \int_{s_1}^{s_2} (-\vec{\sigma}^{(-)} \bullet \vec{n} + \vec{\sigma}^{(+)} \bullet \vec{n}) ds = 0$$



- The local balance at the interface reduces to:

$$[\![\sigma_n(s)]\!] = \vec{\sigma}^{(+)}(s) \bullet \vec{n} - \vec{\sigma}^{(-)}(s) \bullet \vec{n} = 0, \; s \in \Gamma$$

Boundary conditions

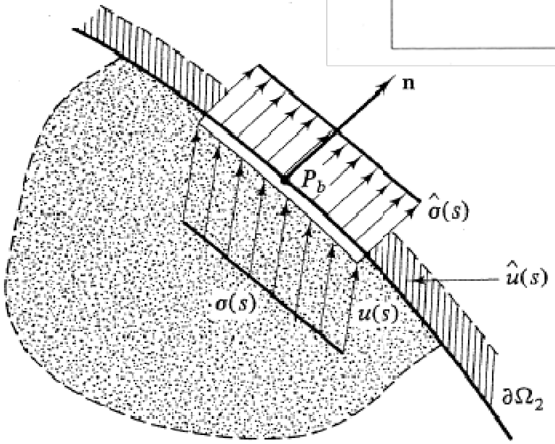


- Let us assume that in the boundary $\Gamma_q \equiv \partial\Omega_2$ we apply natural boundary conditions:

$$\sigma_n(s) \equiv \vec{\sigma}(s) \bullet \vec{n}(s) = \widehat{\sigma}(s) =$$

$$= p(s)(u(s) - \hat{u}(s)),$$

$$s \in \partial\Omega_2$$

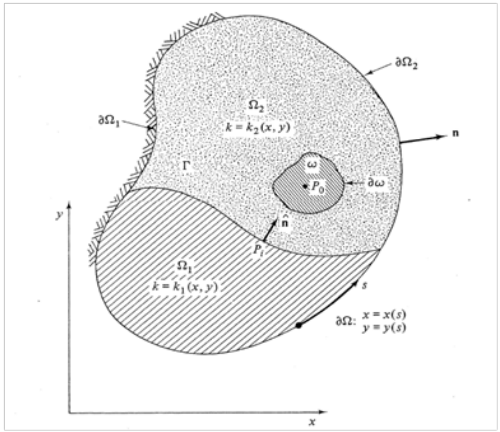


- On the boundary $\Gamma_u \equiv \partial\Omega_1$, we assume essential boundary conditions: $u(s) = \hat{u}(s), s \in \partial\Omega_1$

Summary of the problem of interest

- What we are given:
  - $\partial\Omega_1, \partial\Omega_2$ and the interface Γ
  - The source $f = f(x,y) \; in \; \Omega_i, \; i=1,2$
  - The material moduli $k_i = k_i(x,y), (x,y) \in \Omega_i, \; i=1,2$ $\quad b_i = b_i(x,y), (x,y) \in \Omega_i, \; i=1,2$
  - The boundary coefficients $p(s), \hat{u}(s) \; in \; \partial\Omega_2$

    or $\hat{\sigma}(s) \; in \; \partial\Omega_2$



- With this data we want to compute *u(x,y)* in Ω:

$$-\vec{\nabla} \bullet (k(x,y)\vec{\nabla} u(x,y)) + b(x,y)u(x,y) = f(x,y), (x,y) \in \Omega_i, i=1,2$$

$$-k(s)\frac{\partial u(s)}{\partial n} = p(s)[u(s) - \hat{u}(s)], \; s \in \partial\Omega_2 \; or \qquad -k(s)\frac{\partial u(s)}{\partial n} = \hat{\sigma}(s), \; s \in \partial\Omega_2$$

$$u(s) = \hat{u}(s), \; s \in \partial\Omega_1 \qquad [\![k\vec{\nabla} u \bullet \vec{n}]\!] = 0, \; s \in \Gamma$$

A note regarding the boundary conditions

- The special case in which *b=0* and when only natural BC of the form $-k(s)\frac{\partial u(s)}{\partial n} = \widehat{\sigma}(s),\, s \in \partial\Omega_2$ is applied and when $\partial\Omega_2 \equiv \partial\Omega$, needs special attention:
  - The solution *u(x,y)* can only be determined up to a constant
  - For *u* to exist, the following compatibility condition needs to be satisfied:

$$\int_{\Omega} f dx dy = \int_{\partial\Omega} \widehat{\sigma} ds$$

Variational problem: Weak form

- We start by multiplying the residual *r(x,y)* by a sufficient smooth function *w(x,y)* and integrate over each domain in which *rw* is smooth and set the resulting weighted average equal to zero:

$$r(x,y) = -\vec{\nabla} \bullet (k(x,y)\vec{\nabla}u(x,y)) + b(x,y)u(x,y) - f(x,y)$$

$$\int_{\Omega_1} \left[-\vec{\nabla} \bullet (k(x,y)\vec{\nabla}u(x,y)) + b(x,y)u(x,y) - f(x,y) \right] w(x,y)dxdy +$$

$$\int_{\Omega_2} \left[-\vec{\nabla} \bullet (k(x,y)\vec{\nabla}u(x,y)) + b(x,y)u(x,y) - f(x,y) \right] w(x,y)dxdy = 0$$

- We need to integrate by parts the first terms in each of these integrals

Variational problem

$$\int_{\Omega_1}\left[-\vec{\nabla}\bullet(k(x,y)\vec{\nabla}u(x,y))+b(x,y)u(x,y)-f(x,y)\right]w(x,y)dxdy+$$

$$\int_{\Omega_2}\left[-\vec{\nabla}\bullet(k(x,y)\vec{\nabla}u(x,y))+b(x,y)u(x,y)-f(x,y)\right]w(x,y)dxdy=0$$

- Integration by parts gives:

$$\int_{\Omega_1}\left[k(x,y)\vec{\nabla}u(x,y)\bullet\vec{\nabla}w(x,y)+b(x,y)u(x,y)w(x,y)-f(x,y)w(x,y)\right]dxdy+$$

$$\int_{\Omega_2}\left[k(x,y)\vec{\nabla}u(x,y)\bullet\vec{\nabla}w(x,y)+b(x,y)u(x,y)w(x,y)-f(x,y)w(x,y)\right]dxdy$$

$$-\int_{\Omega_1}\vec{\nabla}\bullet(wk\vec{\nabla}u)dxdy-\int_{\Omega_2}\vec{\nabla}\bullet(wk\vec{\nabla}u)dxdy=0$$

- The last 2 integrals can be simplified with use of the divergence theorem.

Variational problem

$$\int_{\Omega_1} \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) - f(x,y)w(x,y) \right] dxdy +$$

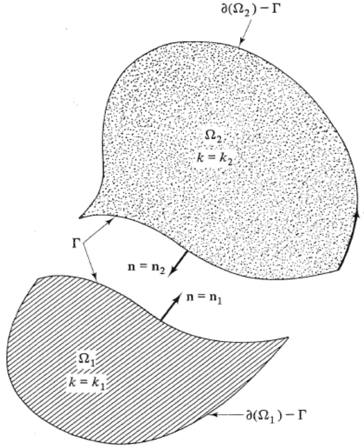
$$\int_{\Omega_2} \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) - f(x,y)w(x,y) \right] dxdy$$

$$- \int_{\underbrace{\partial(\Omega_1)}_{\text{Boundary of the domain } \Omega_1}} k\frac{\partial u}{\partial n} w ds - \int_{\underbrace{\partial(\Omega_2)}_{\text{Boundary of the domain } \Omega_2}} k\frac{\partial u}{\partial n} w ds = 0$$

- We can separate integration over Γ in the last 2 terms:

$$-\int_{\partial(\Omega_1)} k\frac{\partial u}{\partial n} w ds - \int_{\partial(\Omega_2)} k\frac{\partial u}{\partial n} w ds = -\int_{\partial(\Omega_1)-\Gamma} k\frac{\partial u}{\partial n} w ds - \int_{\partial(\Omega_2)-\Gamma} k\frac{\partial u}{\partial n} w ds$$

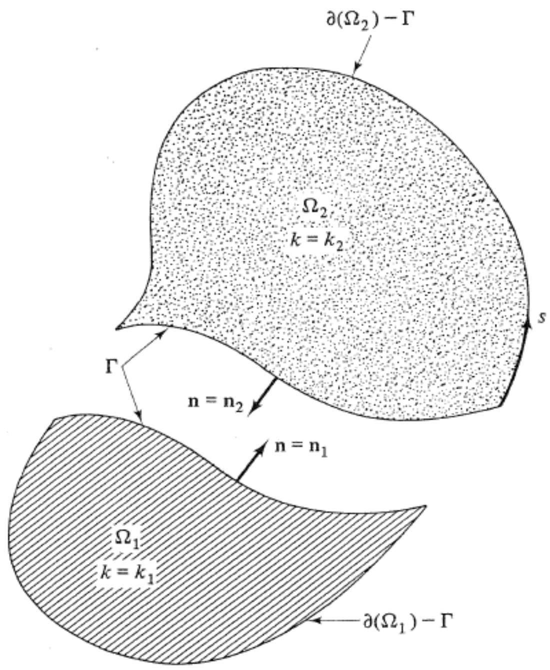
$$+ \underbrace{\int_{\Gamma} (-k\frac{\partial u}{\partial n})_1 w ds}_{\text{evaluated in region 1}} + \underbrace{\int_{\Gamma} (-k\frac{\partial u}{\partial n})_2 w ds}_{\text{evaluated in region 2}}$$



Variational problem

$$-\int_{\partial(\Omega_1)} k\frac{\partial u}{\partial n}wds-\int_{\partial(\Omega_2)} k\frac{\partial u}{\partial n}wds=-\int_{\partial(\Omega_1)-\Gamma} k\frac{\partial u}{\partial n}wds-\int_{\partial(\Omega_2)-\Gamma} k\frac{\partial u}{\partial n}wds+\underbrace{\int_{\Gamma}(-k\frac{\partial u}{\partial n})_1 wds}_{evaluated\ in\ region\ 1}+\underbrace{\int_{\Gamma}(-k\frac{\partial u}{\partial n})_2 wds}_{evaluated\ in\ region\ 2}$$

- Note that the outward normal n_1 to region Ω_1 is the negative of n_2 at each point on Γ.



$$\underbrace{\int_{\Gamma}(-k\frac{\partial u}{\partial n})_1 wds}_{evaluated\ in\ region\ 1}+\underbrace{\int_{\Gamma}(-k\frac{\partial u}{\partial n})_2 wds}_{evaluated\ in\ region\ 2}=\int_{\Gamma}(-k^{(+)}\frac{\partial u^{(+)}}{\partial n}+k^{(-)}\frac{\partial u^{(-)}}{\partial n})wds$$

Variational problem

$$\underbrace{\int_{\Gamma} (-k\frac{\partial u}{\partial n})_1 w ds}_{evaluated\ in\ region\ 1} + \underbrace{\int_{\Gamma} (-k\frac{\partial u}{\partial n})_2 w ds}_{evaluated\ in\ region\ 2} = \int_{\Gamma} (-k^{(+)}\frac{\partial u^{(+)}}{\partial n} + k^{(-)}\frac{\partial u^{(-)}}{\partial n}) w ds$$

- The last integral is nothing else but: $w[\![\sigma_n(s)]\!]$, which is 0!
- Finally, we can summarize our weak form as:

$$\int_{\Omega_1} \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) - f(x,y)w(x,y) \right] dxdy +$$

$$\int_{\Omega_2} \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) - f(x,y)w(x,y) \right] dxdy$$

$$-\int_{\partial(\Omega_1)-\Gamma} k\frac{\partial u}{\partial n} w ds - \int_{\partial(\Omega_2)-\Gamma} k\frac{\partial u}{\partial n} w ds = 0 \Rightarrow$$

$$\int_{\Omega} \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) - f(x,y)w(x,y) \right] dxdy - \int_{\partial\Omega} k\frac{\partial u}{\partial n} w ds = 0$$

Variational problem

$$\int_{\Omega}\left[k(x,y)\vec{\nabla}u(x,y)\bullet\vec{\nabla}w(x,y)+b(x,y)u(x,y)w(x,y)-f(x,y)w(x,y)\right]dxdy-\int_{\underbrace{\partial\Omega}_{\substack{\text{Boundary of the}\\ \text{whole domain }\Omega}}} k\frac{\partial u}{\partial n}wds=0$$

- Substitution of the natural boundary condition

$$-k(s)\frac{\partial u(s)}{\partial n}=p(s)[u(s)-\hat{u}(s)],\ s\in\Gamma_q\equiv\partial\Omega_2$$ gives:

$$\int_{\Omega}\left[k(x,y)\vec{\nabla}u(x,y)\bullet\vec{\nabla}w(x,y)+b(x,y)u(x,y)w(x,y)-f(x,y)w(x,y)\right]dxdy+\int_{\partial\Omega_2} p(u-\hat{u})wds=0$$

for all admissible functions $w(x,y)$ $\quad (w=0 \text{ on } \partial\Omega_1, \text{since we require } u(s)=\hat{u}(s),\ s\in\partial\Omega_1)$

- We write:

$$\int_{\partial\Omega_2} p(u-\hat{u})wds=\int_{\partial\Omega_2} puwds-\int_{\partial\Omega_2}\underbrace{p\hat{u}}_{\gamma}wds=\int_{\partial\Omega_2} puwds-\int_{\partial\Omega_2}\gamma wds$$

Space of admissible functions: $H^1(\Omega)$

- Finally the weak form looks like:

$$\int_{\Omega}\left[k(x,y)\vec{\nabla}u(x,y)\bullet\vec{\nabla}w(x,y)+b(x,y)u(x,y)w(x,y)\right]dxdy+\int_{\partial\Omega_2} puwds=\int_{\Omega} f\,wdxdy+\int_{\partial\Omega_2}\gamma wds$$

$$\textit{for all admissible functions } w(x,y),\ w=0 \textit{ on } \partial\Omega_1$$

- What is the class of admissible functions?

$$\int_{\Omega}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+w^2\right]dxdy<\infty$$

- We indicate this space of functions as $H^1(\Omega)$,
  - 1 reflecting that first derivatives are square integrable
  - Ω indicates the domain over which these functions are defined.

Summary of weak problem

- Find a function $u(x,y) \in H^1(\Omega)$, such that $u(s) = \hat{u}(s),\, s \in \partial\Omega_1$ and the following holds:

$$\int_\Omega \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) \right] dxdy + \int_{\partial\Omega_2} puwds = \int_\Omega fwdxdy + \int_{\partial\Omega_2} \gamma wds$$

$$\textit{for all functions } w(x,y) \in H^1(\Omega) \textit{ with } w = 0 \textit{ on } \partial\Omega_1.$$

- We can repeat the above weak statement using matrix notation as follows (more appropriate for FEM implementation)

Find $u(x,y) \in H^1(\Omega)$, $\quad u(s) = \hat{u}(s),\, s \in \partial\Omega_1$, such that

$\forall w(x,y) \in H^1(\Omega) \textit{ with } w = 0 \textit{ on } \partial\Omega_1$ the following holds:

$$\int_\Omega \left[\underbrace{(\nabla w)^T}_{1x2} \underbrace{k(x,y)}_{1x1} \underbrace{\nabla u}_{2x1} + b(x,y)u(x,y)w(x,y) \right] dxdy + \int_{\partial\Omega_2} puwds = \int_\Omega fwdxdy + \int_{\partial\Omega_2} \gamma wds$$

Matrix form of the weak statement

Find $u(x,y) \in H^1(\Omega), \quad u(s) = \hat{u}(s), s \in \partial\Omega_1$, such that

$\forall w(x,y) \in H^1(\Omega)$ *with* $w = 0$ *on* $\partial\Omega_1$ the following holds:

$$\int_\Omega \left[\underbrace{(\nabla w)^T}_{1x2} \underbrace{k(x,y)}_{1x1} \underbrace{\nabla u}_{2x1} + b(x,y)u(x,y)w(x,y) \right] dxdy + \int_{\partial\Omega_2} puwds = \int_\Omega fwdxdy + \int_{\partial\Omega_2} \gamma wds$$

- The gradient operator here is defined as a column vector:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}, (\nabla w)^T = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}$$

- We will be interchanging the 2 notations here hoping that from the context you know which notation is implied.

Appendices

Some extra slides are included here with:

- ✓ mathematics background,
- ✓ proof of equivalence of weak and strong formulations,
- ✓ Examples of deriving weak forms for 2D BVP,
- ✓ Including anisotropy (e.g. generalized Fourier's law), etc.

Equivalence between the weak and strong problems

- Consider the following simplified BVP and the corresponding weak form:

$$\vec{\nabla} \bullet (\vec{q}(x, y)) = f(x, y), (x, y) \in \Omega$$

$$where : \vec{q} = -k\vec{\nabla}u, \quad -k(s)\frac{\partial u(s)}{\partial n} \equiv \vec{q} \cdot \vec{n} = \bar{q} \; on \; \Gamma_q, \; and \; u = \bar{u} \; on \; \Gamma_u, \; with \; \Gamma_q \cup \Gamma_u \equiv \Gamma (\equiv \partial\Omega)$$

- The weak form is: Find $u(x, y) \in H^1(\Omega) : \forall w \in H^1(\Omega) \; with \; w = 0 \; on \; \Gamma_u :$

$$\int_\Omega \vec{q} \bullet \vec{\nabla} w dxdy + \int_\Omega f w dxdy = \int_{\Gamma_q} \bar{q} w d\Gamma$$

- We want to show that this weak form is equilavent to the strong problem (recover the PDE + natural BCs)

Equivalence between the weak and strong problems

$$\int_{\Omega} \vec{q} \bullet \vec{\nabla} w dxdy + \int_{\Omega} fwdxdy = \int_{\Gamma_q} \overline{q} wd\Gamma$$

- Integration by parts of the first term gives the following:

$$\oint_{\Gamma} w\vec{q} \bullet \vec{n} d\Gamma - \int_{\Omega} w\vec{\nabla} \bullet \vec{q} d\Omega + \int_{\Omega} fwdxdy = \int_{\Gamma_q} \overline{q} wd\Gamma \Rightarrow$$

$$\int_{\Gamma_u} w\vec{q} \bullet \vec{n} d\Gamma + \int_{\Gamma_q} w\vec{q} \bullet \vec{n} d\Gamma - \int_{\Omega} w\left(\vec{\nabla} \bullet \vec{q} - f\right) d\Omega - \int_{\Gamma_q} \overline{q} wd\Gamma = 0 \Rightarrow$$

$$\int_{\Gamma_q} w\left(\vec{q} \bullet \vec{n} - \overline{q}\right) d\Gamma + \int_{\Gamma_u} w\vec{q} \bullet \vec{n} d\Gamma - \int_{\Omega} w\left(\vec{\nabla} \bullet \vec{q} - f\right) d\Omega = 0 \ \forall w \in H^1(\Omega) \ with \ w = 0 \ on \ \Gamma_u$$

Equivalence between the weak and strong problems

$$\int_{\Gamma_q} w\left(\vec{q}\bullet\vec{n}-\overline{q}\right)d\Gamma + \int_{\Gamma_u} w\vec{q}\bullet\vec{n}d\Gamma - \int_{\Omega} w\left(\vec{\nabla}\bullet\vec{q}-f\right)d\Omega = 0 \ \forall w \in H^1(\Omega) \ with \ w=0 \ on \ \Gamma_u$$

- Deriving the PDE first: Select *w* as follows:

$$w=\psi(x)\left(\vec{\nabla}\bullet\vec{q}-f\right), with\ \psi=0\ on\ \Gamma\ and\ \psi>0\ on\ \Omega$$

- The first equation above then results in:

$$\int_{\Omega}\psi\left(\vec{\nabla}\bullet\vec{q}-f\right)^2 d\Omega=0 \Rightarrow \vec{\nabla}\bullet\vec{q}-f=0\ in\ \Omega$$

Equivalence between the weak and strong problems

$$\int_{\Gamma_q} w\left(\vec{q}\bullet\vec{n}-\overline{q}\right)d\Gamma + \int_{\Gamma_u} w\vec{q}\bullet\vec{n}d\Gamma = 0\ \forall w \in H^1(\Omega)\ with\ w = 0\ on\ \Gamma_u$$

- We next derive the natural BC: Select *w* as follows:

$$w = \zeta(x)\left(\vec{q}\bullet\vec{n}-\overline{q}\right),\ with\ \zeta = 0\ on\ \Gamma_u\ and\ \zeta > 0\ on\ \Gamma_q$$

- The first equation above then results in:

$$\int_{\Gamma_q} \zeta\left(\vec{q}\bullet\vec{n}-\overline{q}\right)^2 d\Gamma = 0 \Rightarrow \vec{q}\bullet\vec{n}-\overline{q} = 0\ on\ \Gamma_q$$

Generalized Fourier's law

- In general, the flux $\vec{q}$ and temperature gradient are related with the anisotropic (generalized) Fourier's law:

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = -\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = -\underbrace{[D]}_{2x2\ conductivity\ matrix} \underbrace{\vec{\nabla} u}_{2x1}$$

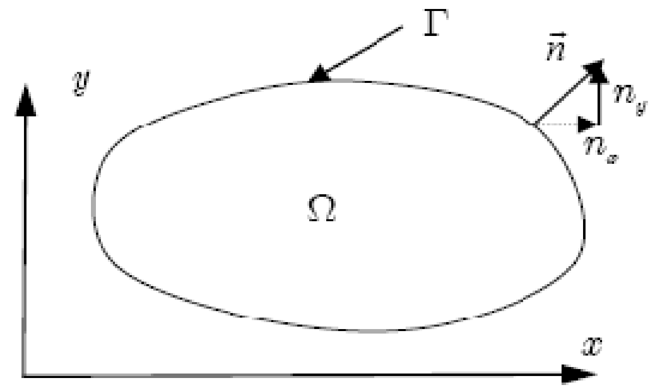
- For isotropic materials $[D] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k \underbrace{[I]}_{2x2\ identity\ matrix}$

- For a 2D BVP with prescribed flux on Γ_q and temperature on Γ_u and a source term *f*, can you show that the weak form in matrix notation is: Find

$$u(x,y) \in H^1(\Omega) : \forall w \in H^1(\Omega)\ with\ w = 0\ on\ \Gamma_u : \int_\Omega \underbrace{\left(\vec{\nabla} w\right)^T}_{1x2} \underbrace{[D]}_{2x2} \underbrace{\vec{\nabla} u}_{2x1} dxdy = -\int_{\Gamma_q} \bar{q} w d\Gamma + \int_\Omega f w dxdy$$

Appendix: Green's theorem

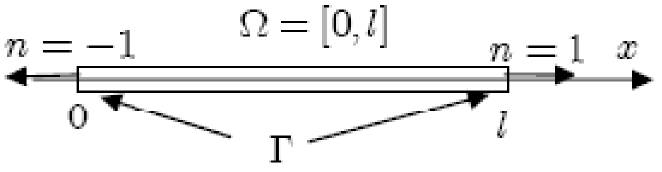
- If $u(x,y) \in C^0$ integrable, then:



$$\int_\Omega \vec{\nabla} u \, d\Omega = \oint_\Gamma u \vec{n} \, d\Gamma$$

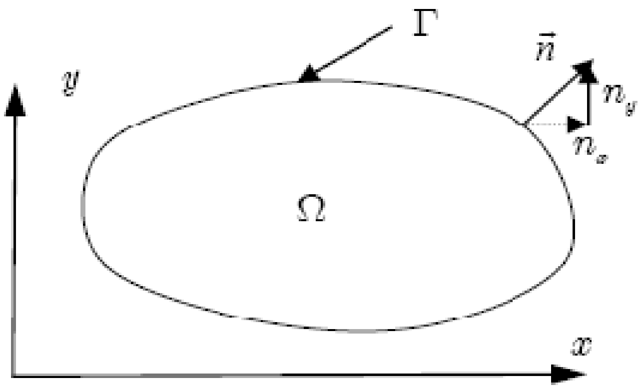
- Recall the one-dimensional version of this (where the boundary is only 2 points):

$$\int_\Omega \frac{du}{dx} dx = un|_\Gamma = u(x=L) - u(x=0)$$



Appendix: Divergence theorem

- If $\vec{q}(x,y) \in C^0$ integrable, then:



$$\int_{\Omega} \vec{\nabla} \bullet \vec{q} d\Omega = \oint_{\Gamma} \vec{q} \bullet \vec{n} d\Gamma$$

- Note that: $\vec{\nabla} \bullet \vec{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$

- Explicilty, we can write the divergence theorem as:

$$\int_{\Omega} \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) d\Omega = \oint_{\Gamma} \left(q_x n_x + q_y n_y \right) d\Gamma$$

Appendix: Green's theorem

- This is the analogue of integration by parts in 1D:

$$\int_{\Omega} w\vec{\nabla}\bullet\vec{q}\,d\Omega = \oint_{\Gamma} w\vec{q}\bullet\vec{n}\,d\Gamma - \int_{\Omega} \vec{\nabla}w\bullet\vec{q}\,d\Omega$$

- The proof of this is simple if you notice that:

$$\vec{\nabla}\bullet(w\vec{q}) = \frac{\partial}{\partial x}(wq_x) + \frac{\partial}{\partial y}(wq_y) = \frac{\partial w}{\partial x}q_x + \frac{\partial q_x}{\partial x}w + \frac{\partial w}{\partial y}q_y + \frac{\partial q_y}{\partial y}w \Rightarrow$$

$$\vec{\nabla}\bullet(w\vec{q}) = w\vec{\nabla}\bullet\vec{q} + \vec{\nabla}w\bullet\vec{q}$$

- This together with the divergence theorem give:

$$\int_{\Omega} w\vec{\nabla}\bullet\vec{q}\,d\Omega = \int_{\Omega} \vec{\nabla}\bullet(w\vec{q})\,d\Omega - \int_{\Omega} \vec{\nabla}w\bullet\vec{q}\,d\Omega = \oint_{\Gamma} w\vec{q}\bullet\vec{n}\,d\Gamma - \int_{\Omega} \vec{\nabla}w\bullet\vec{q}\,d\Omega$$

Appendix: Green's theorem in 1D

$$\int_\Omega w\vec{\nabla}\bullet\vec{q}\,d\Omega = \oint_\Gamma w\vec{q}\bullet\vec{n}\,d\Gamma - \int_\Omega \vec{\nabla}w\bullet\vec{q}\,d\Omega$$

- The 1D analogue of Green's theorem is as follows:

$$\int_\Omega w\frac{\partial q_x}{\partial x}\,d\Omega = \oint_\Gamma w q_x n\,d\Gamma - \int_\Omega \frac{\partial w}{\partial x} q_x\,d\Omega$$

or more explicitly as:

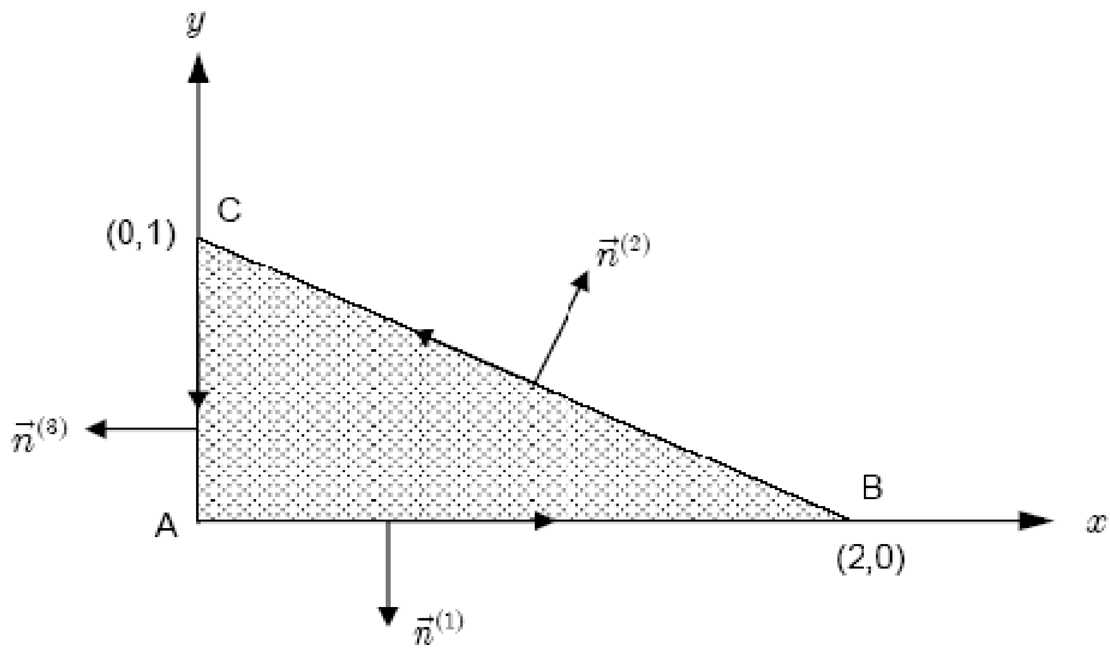
$$\int_\Omega w\frac{\partial q_x}{\partial x}\,dx = wq_x n\,|_\Gamma - \int_\Omega \frac{\partial w}{\partial x} q_x\,dx = \left(wq_x\,|_{x=L} - wq_x\,|_{x=0}\right) - \int_\Omega \frac{\partial w}{\partial x} q_x\,dx$$

Recall that we had used this compact form in 1D to treat boundary conditions on the left (x=0) and the right (x=L) end points in a unified way

Example problem on divergence theorem

- Consider the vector function $\vec{q} = (q_x . q_y) = (3x^2 y + y^3, 3x + y^3),$ defined in the domain shown in the figure below.
- Demonstrate the validity of the divergence theorem, i.e.

$$\int_{\Omega} \vec{\nabla} \bullet \vec{q} d\Omega = \int_{\Gamma} \vec{q} \bullet \vec{n} d\Gamma$$

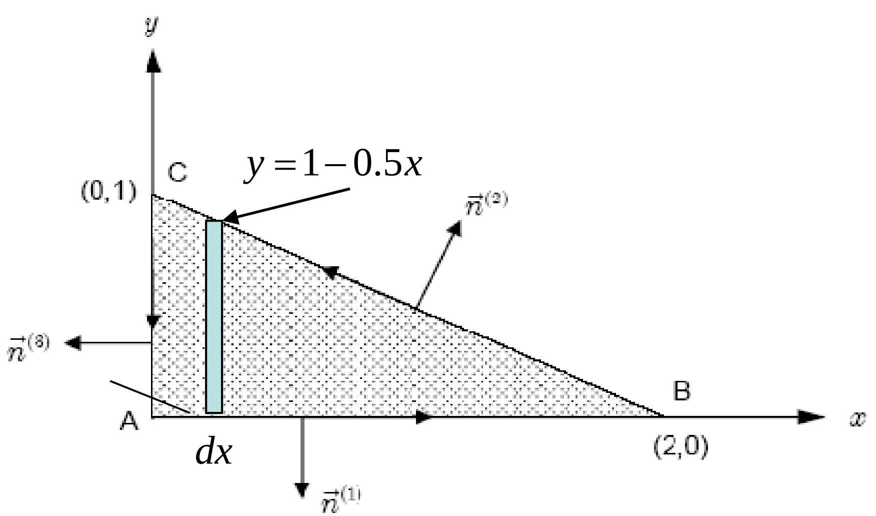


Example problem on divergence theorem

- From $\vec{q} = (q_x . q_y) = (3x^2 y + y^3, 3x + y^3),$ we compute the following: $\vec{\nabla} \bullet \vec{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = \left(6xy\right) + (3y^2) = 6xy + 3y^2$

- Using this, we compute $\int_{\Omega} \vec{\nabla} \bullet \vec{q} d\Omega$ as follows:

$$\int_{\Omega} \vec{\nabla} \bullet \vec{q} d\Omega = \int_0^2 \int_0^{1-0.5x} (6xy + 3y^2) dydx = \int_0^2 [3x(1-0.5x)^2 + (1-0.5x)^3] dx = 1.5$$



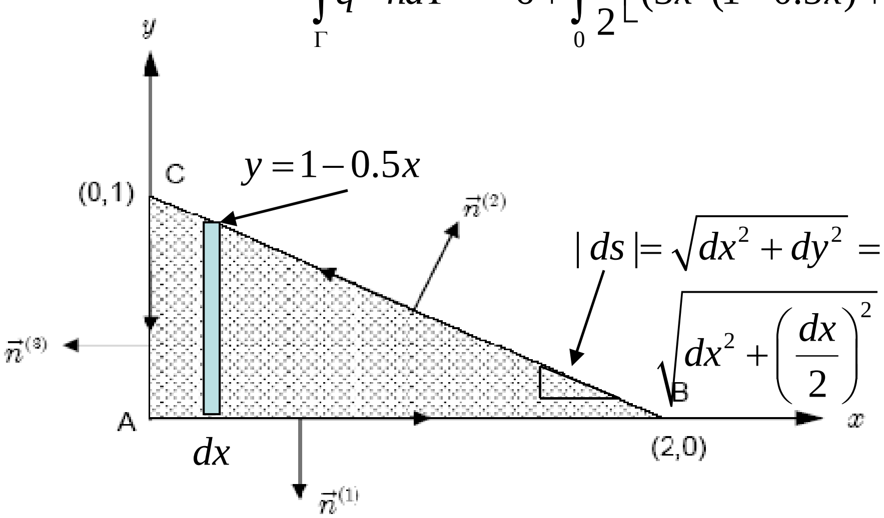
Example problem on divergence theorem

- Also: $\int_{\Gamma}\vec{q}\bullet\vec{n}d\Gamma=\int_{AB}\vec{q}\bullet\underbrace{\vec{n}}_{(0,-1)}\underbrace{d\Gamma}_{dx}+\int_{BC}\vec{q}\bullet\underbrace{\vec{n}}_{\frac{\sqrt{5}}{5}(1,2)}\underbrace{d\Gamma}_{-\frac{\sqrt{5}}{2}dx}+\int_{CA}\vec{q}\bullet\underbrace{\vec{n}}_{(-1,0)}\underbrace{d\Gamma}_{-dy}$

- Note that on segment BC: $ds=-\frac{\sqrt{5}}{2}dx$ and $\vec{n}^{(2)}=\frac{\sqrt{5}}{5}(1,2)$. Thus

$$\int_{\Gamma}\vec{q}\bullet\vec{n}d\Gamma=\int_0^2-(3x+\underbrace{y}_{0}{}^3)dx+\int_2^0\frac{\sqrt{5}}{5}\left[(3x^2\underbrace{y}_{1-0.5x}+y^3)+2(3x+y^3)\right](-\frac{\sqrt{5}}{2}dx)+\int_1^0(-3\underbrace{x}_{0}{}^2y-y^3)(-dy)\Rightarrow$$

$$\int_{\Gamma}\vec{q}\bullet\vec{n}d\Gamma=-6+\int_0^2\frac{1}{2}\left[(3x^2(1-0.5x)+(1-0.5x)^3)+2(3x+(1-0.5x)^3)\right]dx-0.25\Rightarrow$$



$$|ds|=\sqrt{dx^2+dy^2}=\sqrt{dx^2+\left(\frac{dx}{2}\right)^2}=\frac{\sqrt{5}}{2}|dx|$$

$$\int_{\Gamma}\vec{q}\bullet\vec{n}d\Gamma=-6+7.75-0.25$$

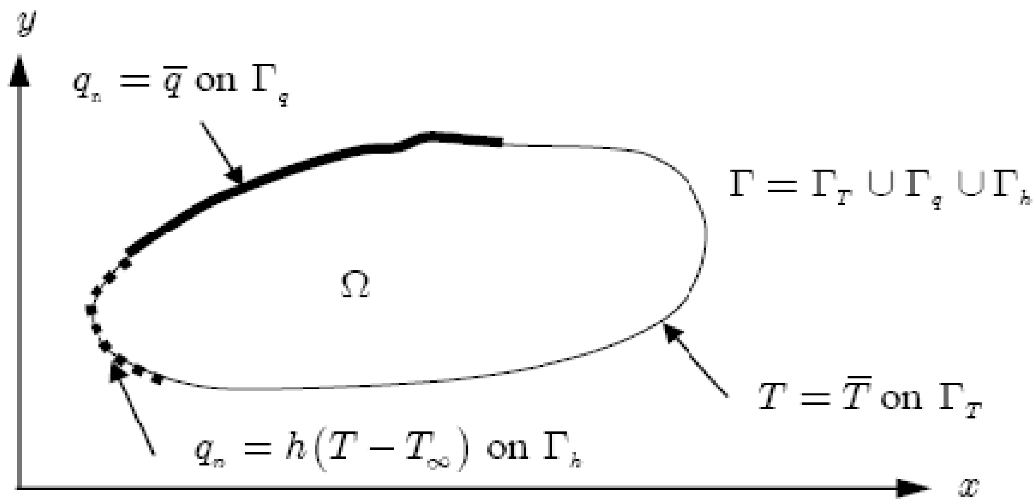
$$\int_{\Gamma}\vec{q}\bullet\vec{n}d\Gamma=-6+7.75-0.25=1.5\Rightarrow$$

$$\int_{\Omega}\vec{\nabla}\bullet\vec{q}d\Omega=\int_{\Gamma}\vec{q}\bullet\vec{n}d\Gamma$$

Example: Deriving the weak form

- Consider isotropic heat conduction in the domain shown with the boundary conditions indicated. Provide a complete weak statement of the problem.

$$-\vec{\nabla} \bullet (k(x,y)\vec{\nabla}T(x,y)) = f(x,y)$$



- We start by multiplying with the weight function *w* and integrating over the domain:

$$\int_\Omega \left[-\vec{\nabla} \bullet (k(x,y)\vec{\nabla}T(x,y)) - f(x,y) \right] w(x,y)\,dxdy = 0, \forall \text{ sufficiently smooth } w$$

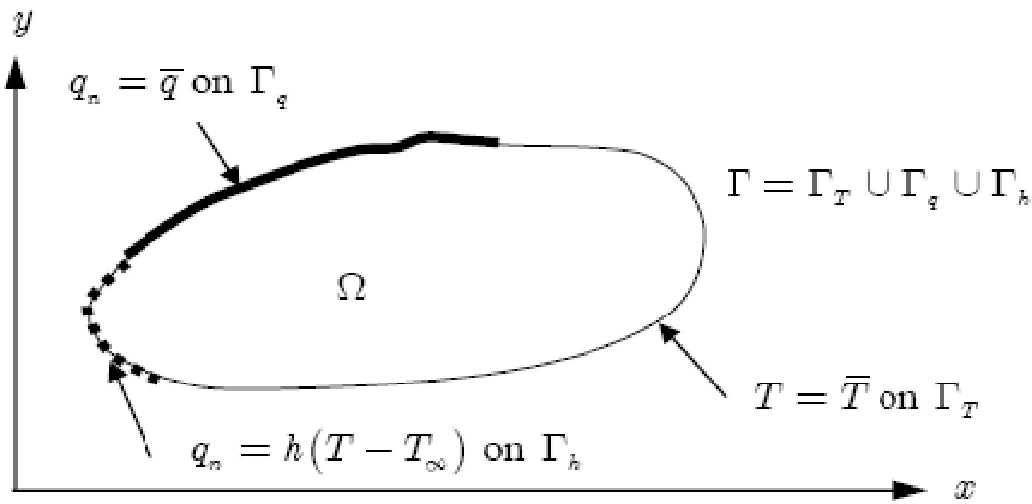
Example: Deriving the weak form

$$\int_\Omega \left[-\vec{\nabla} \bullet (k(x,y)\vec{\nabla}T(x,y)) - f(x,y) \right] w(x,y)\,dxdy = 0, \forall \; sufficiently \; smooth \; w$$

- Integration by parts gives:

$$\int_\Omega \left[k\vec{\nabla}T \bullet \vec{\nabla}w - fw \right] dxdy - \int_{\Gamma_q + \Gamma_h} k \frac{\partial T}{\partial n} w ds = 0, \forall \;\; w \in H^1(\Omega), with \; w = 0 \; on \; \Gamma_T$$

- The boundary term can be further simplified as:



$$\int_{\Gamma_q + \Gamma_h} k \frac{\partial T}{\partial n} w ds = \int_{\Gamma_q} (-\overline{q}) w ds + \int_{\Gamma_h} (-h(T - T_\infty)) w ds = -\int_{\Gamma_q} \overline{q} w ds - \int_{\Gamma_h} hTwds + \int_{\Gamma_h} hT_\infty wds$$

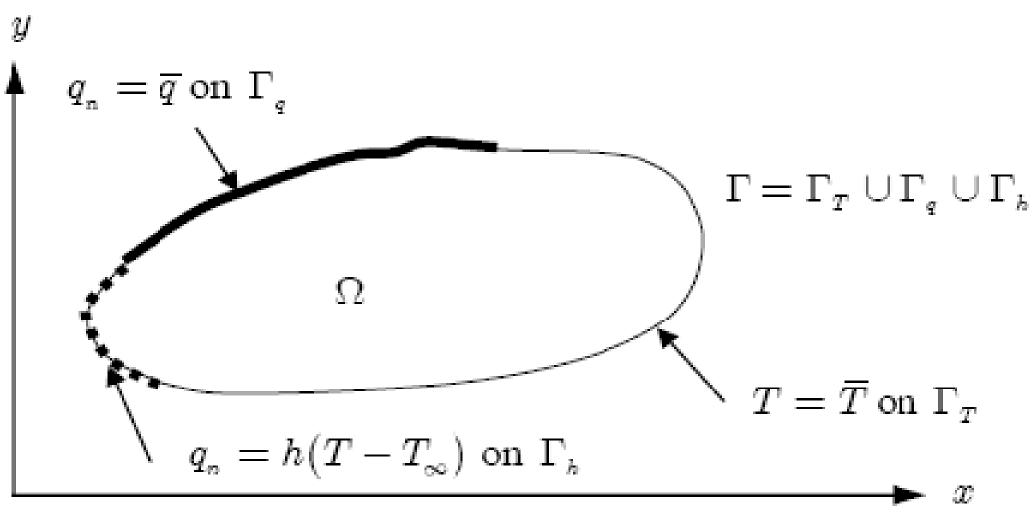
Example: Deriving the weak form

$$\int_{\Omega}\left[k\vec{\nabla}T\bullet\vec{\nabla}w-fw\right]dxdy-\int_{\Gamma_q+\Gamma_h}k\frac{\partial u}{\partial n}wds=0,\forall\ w\in H^1(\Omega), with\ w=0\ on\ \Gamma_T$$

- The final weak form is then:

$$Find\ T(x,y)\in H^1(\Omega), with\ T=\overline{T}\ on\ \Gamma_T\ such\ that$$

$$\int_{\Omega}\left[k\vec{\nabla}T\bullet\vec{\nabla}w-fw\right]dxdy+\int_{\Gamma_h}hTwds=-\int_{\Gamma_q}\overline{q}wds+\int_{\Gamma_h}hT_{\infty}wds,\forall\ w\in H^1(\Omega), with\ w=0\ on\ \Gamma_T$$



In matrix form, we can also write the weak form as:

$$\int_{\Omega}\left[\underbrace{(\nabla w)^T}_{1x2}\underbrace{k}_{1x1}\underbrace{\nabla T}_{2x1}-fw\right]dxdy$$
$$+\int_{\Gamma_h}hTwds=-\int_{\Gamma_q}\overline{q}wds+\int_{\Gamma_h}hT_{\infty}wds$$
$$\forall\ w\in H^1(\Omega), with\ w=0\ on\ \Gamma_T$$