MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

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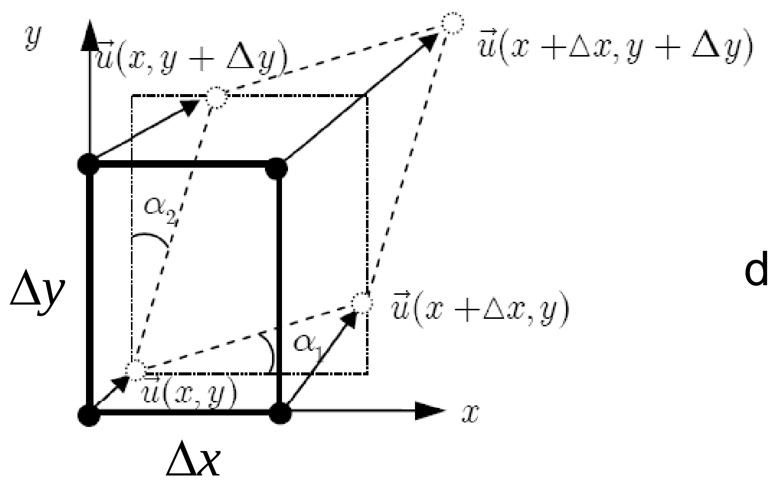
Kinematics in two-dimensions

- We start with the displacement vector at a given point in 2D. It is a vector with *x*- and the *y*-components. We denote it using both matrix and vector notation:

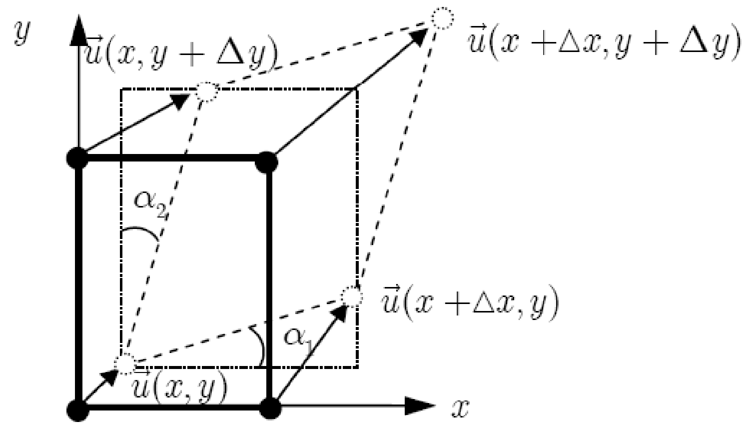
$$u = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad \vec{u} = u_x \vec{i} + u_y \vec{j}$$

- Let us consider a square $\Delta x \Delta y$ (control volume) on the plane before and after deformation as shown:



Can you use this figure to define the strain components?

Extensional strain components



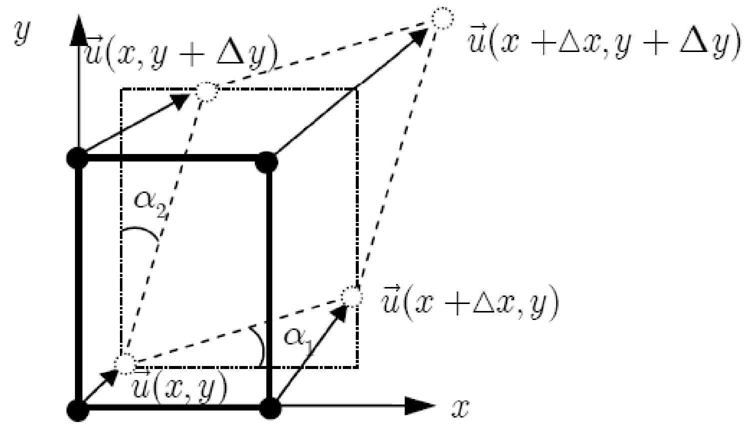
- The extensional strains ε_{xx} and ε_{yy} are defined as:

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x} \equiv \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{u_y(x, y+\Delta y) - u_y(x, y)}{\Delta y} \equiv \frac{\partial u_y}{\partial y}$$

- ε_{xx} and ε_{yy} represent the change in the lengths of the infinitesimal line segments in the x and y directions, Δx and Δy , respectively, divided by the original lengths of the line segments.

Shear strain components



- The shear strain, γ_{xy} , measures the change in angle $\alpha_1 + \alpha_2$ between unit vectors in the x and y directions (in radians)

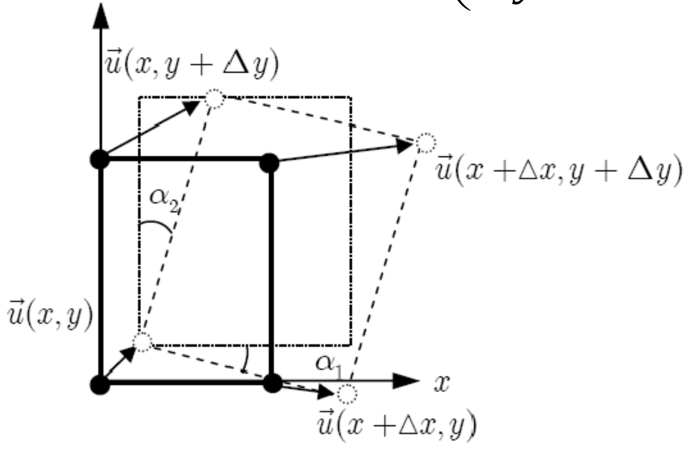
$$\gamma_{xy} = \lim_{\Delta x \to 0} \frac{u_y(x+\Delta x, y) - u_y(x, y)}{\Delta x} + \lim_{\Delta y \to 0} \frac{u_x(x, y+\Delta y) - u_x(x, y)}{\Delta y} \equiv \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

- We often use the engineering shear strain γ_{xy} introduced above, and the tensor shear strain component $\varepsilon_{xy} = ½\, \gamma_{xy}$.

Rotation

- In addition to axial elongations, the control volume also undergoes rotation. The rotation in 2D, denoted by ω_{xy} , is computed by

$$\omega_{xy} \equiv \frac{1}{2}(\alpha_2 - \alpha_1) = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$



- If $\alpha_1 = \alpha_2$, the rotation is zero. For small deformations, the rotation ω_{xy} is small and does not affect the stress calculation.

Strain matrix

- For our finite element calculations, we introduce the following notation for the strains:

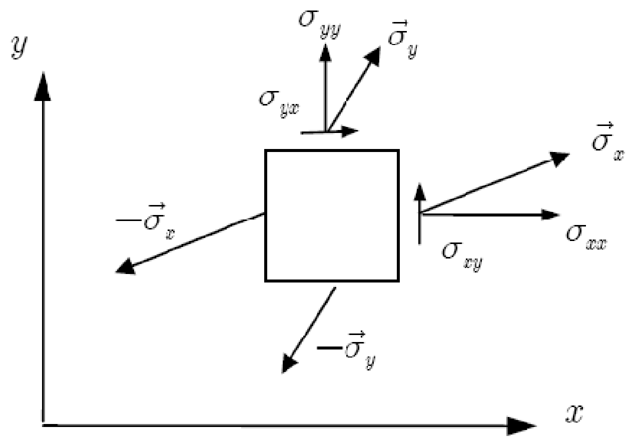
$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}}_{\nabla_s} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \nabla_S u$$

- As shown above, the symmetric gradient matrix ∇_S is defined as:

$$\nabla_S = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

Traction vector and stress components

- The traction (force per unit area) on the plane with the normal vector $\vec{n}$ aligned along the *x*-axis is denoted by $\vec{\sigma}_x$ and its vector form is $\vec{\sigma}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}$
- Similarly, the traction with the outer normal unit vector $\vec{n}$ aligned along the *y* -axis is denoted as $\vec{\sigma}_y$ and its corresponding components are $\vec{\sigma}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}$



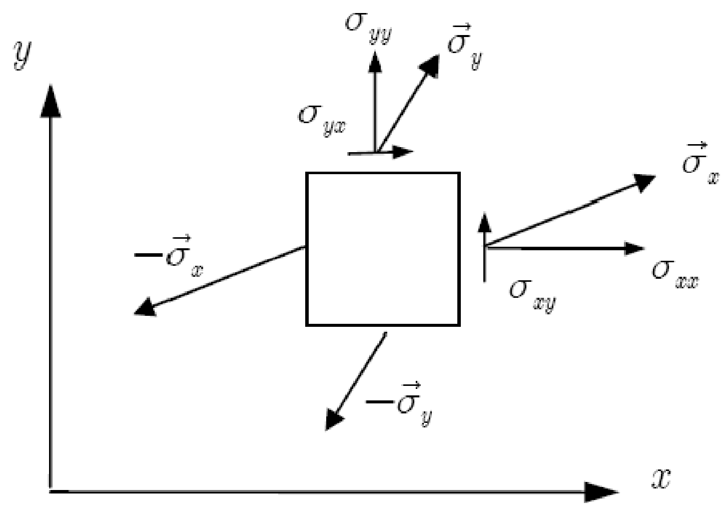
$\vec{\sigma}_x$ and $\vec{\sigma}_y$ are called the stress vectors acting on the planes normal to *x* and *y* directions, respectively.

The stress state in a 2D body is described by two normal stresses σ_{xx} and σ_{yy} and shear stresses σ_{xy} and σ_{yx}.

Can you show that $\sigma_{xy}=\sigma_{yx}$? (moment equilibrium)

Stress components

- Positive stress components act in the positive direction on a positive face.
- The 1st subscript on the stress corresponds to the direction normal to the plane and the 2nd subscript denotes the direction of the force. The normal stresses are often written simply as σ_x and σ_y.



In matrix form, we denote the stress components as

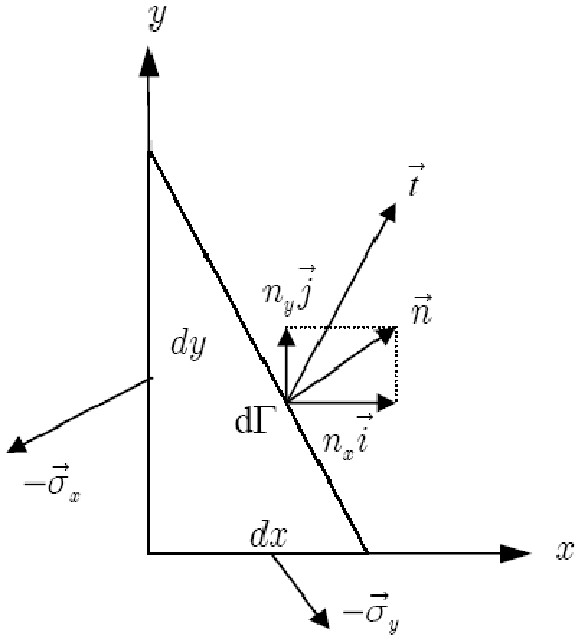
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

or as

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

Traction in an arbitrary surface with normal n

- The stress vectors $\vec{\sigma}_x$ and $\vec{\sigma}_y$ can be used to obtain the tractions on any surface of the body with normal $\vec{n}$. Thus the stresses provide information about tractions on any surface at a point.



The force equilibrium of the triangular body shown requires that:

$$\vec{t}d\Gamma - \vec{\sigma}_x dy - \vec{\sigma}_y dx = 0 \Rightarrow$$

$$\vec{t}d\Gamma - \vec{\sigma}_x n_x d\Gamma - \vec{\sigma}_y n_y d\Gamma = 0 \Rightarrow$$

$$\vec{t} = \vec{\sigma}_x n_x + \vec{\sigma}_y n_y \Rightarrow$$

$$\vec{t} = \left(\sigma_{xx}\vec{i} + \sigma_{xy}\vec{j}\right)n_x + \left(\sigma_{yx}\vec{i} + \sigma_{yy}\vec{j}\right)n_y \Rightarrow$$

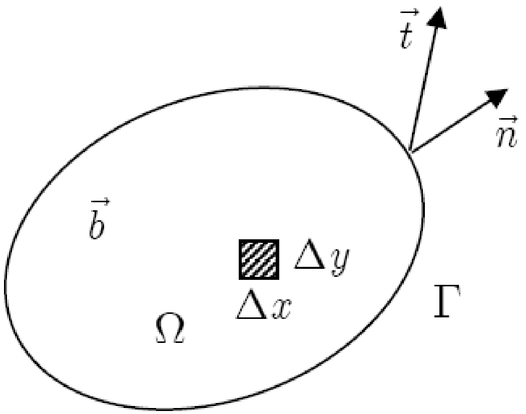
$$\vec{t} = \underbrace{\left(\sigma_{xx}n_x + \sigma_{yx}n_y\right)}_{t_x}\vec{i} + \underbrace{\left(\sigma_{yx}n_x + \sigma_{yy}n_y\right)}_{t_y}\vec{j}$$

In matrix form $$\underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{t} = \underbrace{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}}_{\tau} \underbrace{\begin{bmatrix} n_x \\ n_y \end{bmatrix}}_{n}, \quad t = \tau n$$

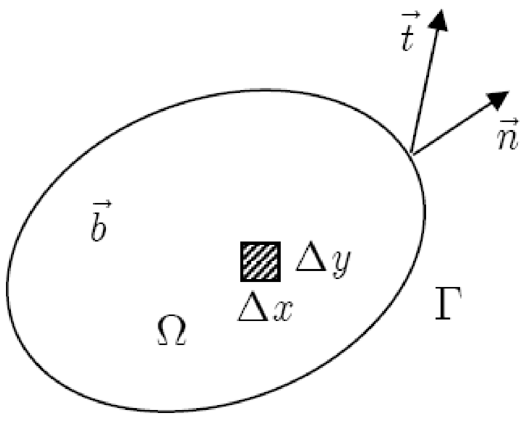
Stress equilibrium

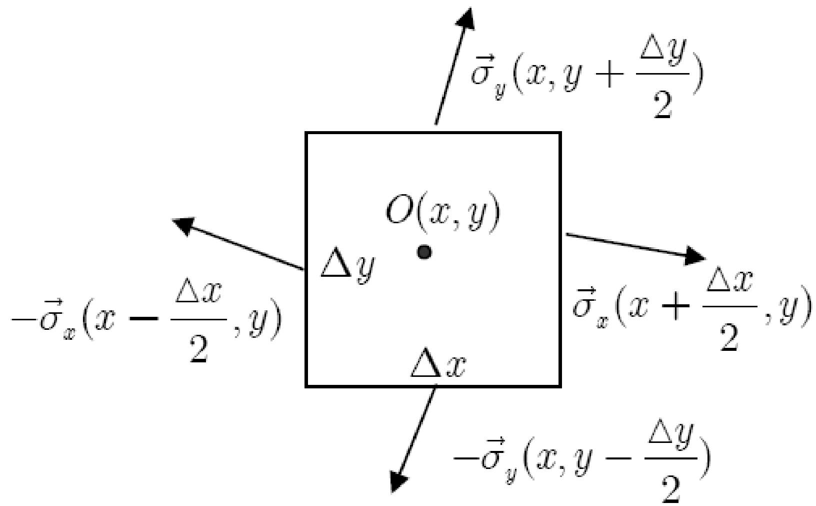
- The forces acting on the body are the traction vector $\vec{t} = t_x \vec{i} + t_y \vec{j}$ along the boundary *Γ* and the body force per unit volume $\vec{b} = b_x \vec{i} + b_y \vec{j}$.

- Examples of the body forces are gravity and magnetic forces, etc. Thermal stresses as we have seen can also been interpreted as body forces.



Stress equilibrium



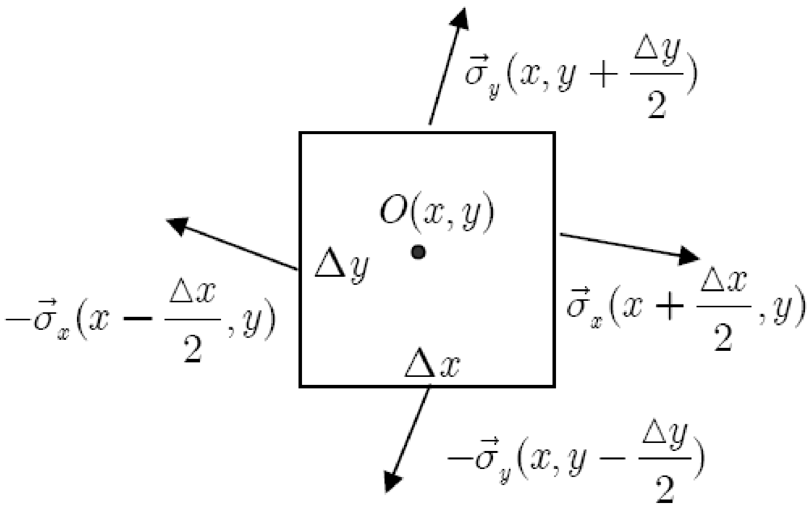


- Consider the equilibrium of the infinitesimal region (on the plane) of unit thickness:

$$-\vec{\boldsymbol{\sigma}}_x(x-\frac{\Delta x}{2},y)\Delta y+\vec{\boldsymbol{\sigma}}_x(x+\frac{\Delta x}{2},y)\Delta y-\vec{\boldsymbol{\sigma}}_y(x,y-\frac{\Delta y}{2})\Delta x+\vec{\boldsymbol{\sigma}}_y(x,y+\frac{\Delta y}{2})\Delta x+\vec{b}(x,y)\Delta x\Delta y=0\Rightarrow$$

$$\frac{\vec{\boldsymbol{\sigma}}_x(x+\frac{\Delta x}{2},y)-\vec{\boldsymbol{\sigma}}_x(x-\frac{\Delta x}{2},y)}{\Delta x}+\frac{\vec{\boldsymbol{\sigma}}_y(x,y+\frac{\Delta y}{2})-\vec{\boldsymbol{\sigma}}_y(x,y-\frac{\Delta y}{2})}{\Delta y}+\vec{b}(x,y)=0 \underbrace{\Rightarrow}_{\Delta x,\Delta y\to 0}$$

Stress equilibrium



$$\frac{\partial \vec{\boldsymbol{\sigma}}_x}{\partial x} + \frac{\partial \vec{\boldsymbol{\sigma}}_y}{\partial y} + \vec{b}(x, y) = 0$$

However: $\vec{\boldsymbol{\sigma}}_x = \boldsymbol{\sigma}_{xx}\vec{i} + \boldsymbol{\sigma}_{xy}\vec{j}$

$$\vec{\boldsymbol{\sigma}}_y = \boldsymbol{\sigma}_{yx}\vec{i} + \boldsymbol{\sigma}_{yy}\vec{j}$$

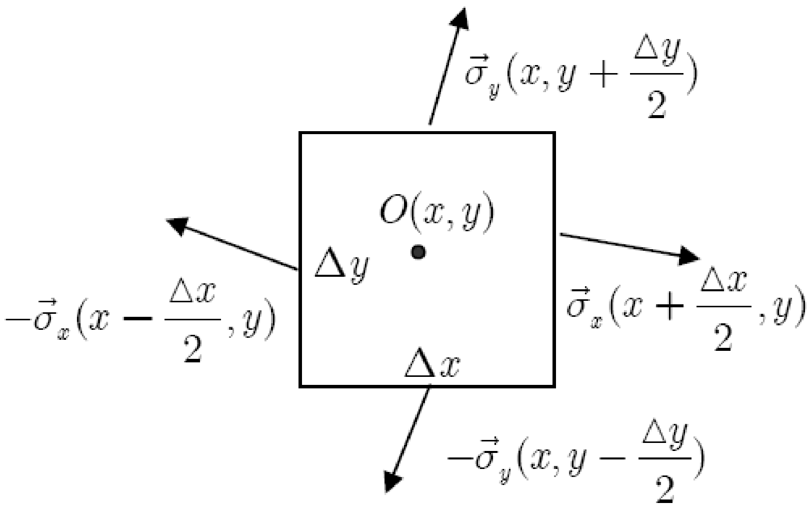
- Combining the above two equations yields the equilibrium equations for the x- and y-directions:

$$\frac{\partial\left(\boldsymbol{\sigma}_{xx}\vec{i} + \boldsymbol{\sigma}_{xy}\vec{j}\right)}{\partial x} + \frac{\partial\left(\boldsymbol{\sigma}_{yx}\vec{i} + \boldsymbol{\sigma}_{yy}\vec{j}\right)}{\partial y} + \vec{b}(x, y) = 0 \Rightarrow \begin{array}{l} \dfrac{\partial \boldsymbol{\sigma}_{xx}}{\partial x} + \dfrac{\partial \boldsymbol{\sigma}_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \boldsymbol{\sigma}_{yx}}{\partial x} + \dfrac{\partial \boldsymbol{\sigma}_{yy}}{\partial y} + b_y = 0 \end{array}$$

or

$$\begin{array}{l} \nabla \bullet \vec{\boldsymbol{\sigma}}_x + b_x = 0 \\ \nabla \bullet \vec{\boldsymbol{\sigma}}_y + b_y = 0 \end{array}$$

Stress equilibrium



$$\frac{\partial \boldsymbol{\sigma}_{xx}}{\partial x} + \frac{\partial \boldsymbol{\sigma}_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \boldsymbol{\sigma}_{yx}}{\partial x} + \frac{\partial \boldsymbol{\sigma}_{yy}}{\partial y} + b_y = 0$$

- We can now re-write the equilibrium equation in matrix form:

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\nabla_s^T} \underbrace{\begin{bmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{bmatrix}}_{\boldsymbol{\sigma}} + \underbrace{\begin{bmatrix} b_x \\ b_y \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0} \Rightarrow \qquad \nabla_s^T \boldsymbol{\sigma} + b = 0$$

Constitutive equations

- Constitutive equation is the relation between stress and strain. Examples include elasticity, viscoelasticity, creep, plasticity, viscoplasticity, etc.
- Here, we discuss linear isotropic elasticity.
- Recall that in 1D, a linear elastic material is governed by the Hooke's law $\sigma = E\varepsilon$, where E is Young's modulus.
- In 2D, the linear relation between the stress and strain matrices can be written as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

where **D** is a 3x3 matrix.

Constitutive equations

$$\boldsymbol{\sigma} = \mathbf{D}\varepsilon$$

- This is the generalized Hooke's law. **D** is a symmetric, positive-definite matrix.

- In 2D, the matrix **D** depends on whether one assumes a plane stress or plane strain condition. These assumptions determine how the model is simplified from a 3D physical body to a 2D model.

Plane strain conditions

- A plane strain model assumes that the body is thick relative to the *xy* -plane in which the model is constructed.

- Consequently, the strain normal to the plane, ε_z is zero and the shear strains that involve angles normal to the plane, γ_{xz} and , γ_{yz} are assumed to vanish.

- When a body is thick, significant stresses can develop on the *z* -faces, in particular the normal stress σ_{zz} can be quite large.

Plane stress

- A plane stress model assumes that the body is thin relative to the dimensions in the *xy* -plane.

- In that case, we assume that no loads are applied on the *z* -faces of the body and that the stress normal to the *xy* -plane, σ_{zz} , vanishes.

- If a body is thin, since the stress σ_{zz} must vanish on the outside surfaces, there is no mechanism for developing a significant nonzero stress σ_{zz} within the body.

Constitutive equations

- Here we assume an isotropic material, i.e. a material whose stress-strain law is independent of the coordinate system.
- For an isotropic material, **D** is the same regardless of the coordinate system. Of course this is not the best approximation for all problems!

Plane stress:

$$D = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & {(1-v)}/{2} \end{bmatrix}$$

Young's modulus *E* and Poisson's ratio *v* *are the only independent material properties for a linear isotropic elastic material*

Plane strain:

$$D = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & {(1-2v)}/{2} \end{bmatrix}$$

Constitutive equations

- For an isotropic elastic material we have only 2 independent material constants: E and v (of course **D** can be written using any two other elastic material constants such as the shear modulus $G=E/2(1+v)$ and the bulk modulus $K=E/3(1-v)$.

- Note that for plane strain, as $v \to 0.5$, **D** becomes infinite.

- A Poisson's ratio of 0.5 corresponds to an incompressible material.

- Modeling incompressible materials for plane strain (and 3D) problems requires special attention in finite element analysis (and special elements).

Strong form of an elasticity problem

- Equilibrium equation: $\nabla_s^T \boldsymbol{\sigma} + b = 0$ or equivalently: $\vec{\nabla} \bullet \vec{\boldsymbol{\sigma}}_x + b_x = 0$

  $\vec{\nabla} \bullet \vec{\boldsymbol{\sigma}}_y + b_y = 0$

- Kinematics equation (strain-displacement relation):

$$\boldsymbol{\varepsilon} = \nabla_s u$$

- Constitutive equation (stress-strain relation):

$$\sigma = \mathbf{D}\varepsilon$$

- Boundary conditions: The portion of the boundary where the traction is prescribed is denoted by Γ_t and the portion of the boundary where the displacement is prescribed Γ_u .

  - ✓ The traction boundary condition is written as $\boldsymbol{\tau} n = \bar{t} \;\; on \;\; \Gamma_t$ or equivalently:

$$\vec{\boldsymbol{\sigma}}_x \bullet \bar{n} = \bar{t}_x, \vec{\boldsymbol{\sigma}}_y \bullet \bar{n} = \bar{t}_y \;\; on \;\; \Gamma_t$$

  - ✓ The displacement boundary condition is written as $\vec{u} = \bar{\vec{u}} \;\; on \;\; \Gamma_u$

Strong form

- Essential boundary condition: The displacement boundary condition is the essential boundary condition satisfied by the displacement field.
- Natural boundary condition: The traction boundary condition is a natural boundary condition.
- The displacement and traction cannot both be prescribed on the same part of the boundary, thus

$$\Gamma_u \cap \Gamma_t = 0.$$

- However, on any portion of the boundary, either the displacement or the traction must be prescribed, so

$$\Gamma_u \cup \Gamma_t = \Gamma$$

Strong form for isotropic linear elasticity

- Find the displacement field $\vec{u}$ on Ω such that:

$$\vec{\nabla}\bullet\vec{\sigma}_x + b_x = 0, \vec{\nabla}\bullet\vec{\sigma}_y + b_y = 0 \quad on \;\; \Omega$$

$$where \;\; \sigma = D\nabla_s u$$

$$with$$

$$\vec{\sigma}_x \bullet \bar{n} = \bar{t}_x, \vec{\sigma}_y \bullet \bar{n} = \bar{t}_y \quad on \;\; \Gamma_t$$

$$\vec{u} = \bar{\vec{u}} \quad on \;\; \Gamma_u$$

Weak form for isotropic linear elasticity

- We pre-multiply the equilibrium equations in *x* and *y* directions and the two natural boundary conditions by the corresponding weight functions and integrate over the corresponding domains as follows:

$$\vec{\nabla}\bullet\vec{\sigma}_x + b_x = 0, \vec{\nabla}\bullet\vec{\sigma}_y + b_y = 0 \; on \;\; \Omega \Rightarrow$$

$$\int_\Omega \left(\vec{\nabla}\bullet\vec{\sigma}_x + b_x\right) w_x d\Omega = 0 \; \forall w_x \in U_0,$$

$$\int_\Omega \left(\vec{\nabla}\bullet\vec{\sigma}_y + b_y\right) w_y d\Omega = 0 \; \forall w_y \in U_0.$$

$$\vec{\sigma}_x \bullet \bar{n} = \bar{t}_x, \vec{\sigma}_y \bullet \bar{n} = \bar{t}_y \;\; on \;\; \Gamma_t \Rightarrow \int_{\Gamma_t} w_x \left(\vec{\sigma}_x \bullet \bar{n} - \bar{t}_x\right) d\Gamma = 0 \; \forall w_x \in U_0,$$

$$\int_{\Gamma_t} w_y \left(\vec{\sigma}_y \bullet \bar{n} - \bar{t}_y\right) d\Gamma = 0 \; \forall w_y \in U_0.$$

Weak form for isotropic linear elasticity

$$\int_{\Omega} \left(\vec{\nabla} \bullet \vec{\boldsymbol{\sigma}}_x + b_x \right) w_x d\Omega = 0 \ \forall w_x \in U_0,$$

$$\int_{\Omega} \left(\vec{\nabla} \bullet \vec{\boldsymbol{\sigma}}_y + b_y \right) w_y d\Omega = 0 \ \forall w_y \in U_0.$$

- Apply Green's theorem (integration by parts) to the first term in each of these equations and account for $w_x = w_y = 0 \ on \ \Gamma_u$

$$\int_{\Gamma_t} w_x \vec{\boldsymbol{\sigma}}_x \bullet \vec{n} d\Gamma - \int_{\Omega} \vec{\nabla} w_x \bullet \vec{\boldsymbol{\sigma}}_x d\Omega + \int_{\Omega} b_x w_x d\Omega = 0 \ \forall w_x \in U_0,$$

$$\int_{\Gamma_t} w_y \vec{\boldsymbol{\sigma}}_y \bullet \vec{n} d\Gamma - \int_{\Omega} \vec{\nabla} w_y \bullet \vec{\boldsymbol{\sigma}}_y d\Omega + \int_{\Omega} b_y w_y d\Omega = 0 \ \forall w_y \in U_0.$$

- Adding the two equations gives:

$$\int_{\Omega} \left(\vec{\nabla} w_x \bullet \vec{\boldsymbol{\sigma}}_x + \vec{\nabla} w_y \bullet \vec{\boldsymbol{\sigma}}_y \right) d\Omega = \int_{\Gamma_t} \vec{w} \bullet \vec{t} d\Gamma + \int_{\Omega} \vec{w} \bullet \vec{b} d\Omega \ \forall w_x \in U_0$$

Weak form for isotropic linear elasticity

$$\int_{\Omega}\left(\vec{\nabla} w_x \bullet \vec{\boldsymbol{\sigma}}_x + \vec{\nabla} w_y \bullet \vec{\boldsymbol{\sigma}}_y\right) d\Omega = \int_{\Gamma_t} \vec{w} \bullet \vec{t}\, d\Gamma + \int_{\Omega} \vec{w} \bullet \vec{b}\, d\Omega \quad \forall w_x \in U_0$$

- The left hand side can be simplified by noticing that:

$$\vec{\nabla} w_x \bullet \vec{\boldsymbol{\sigma}}_x + \vec{\nabla} w_y \bullet \vec{\boldsymbol{\sigma}}_y = \frac{\partial w_x}{\partial x}\boldsymbol{\sigma}_{xx} + \frac{\partial w_x}{\partial y}\boldsymbol{\sigma}_{xy} + \frac{\partial w_y}{\partial x}\boldsymbol{\sigma}_{xy} + \frac{\partial w_y}{\partial y}\boldsymbol{\sigma}_{yy} =$$

$$= \begin{bmatrix} \frac{\partial w_x}{\partial x} & \frac{\partial w_y}{\partial y} & \frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{bmatrix} = \left(\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\nabla_S} \begin{bmatrix} w_x \\ w_y \end{bmatrix} \right)^T \begin{bmatrix} \boldsymbol{\sigma}_{xx} \\ \boldsymbol{\sigma}_{yy} \\ \boldsymbol{\sigma}_{xy} \end{bmatrix} = \left(\nabla_S w\right)^T \boldsymbol{\sigma}$$

Weak form for isotropic linear elasticity

$$\int_{\Omega}\left(\vec{\nabla} w_x \bullet \vec{\boldsymbol{\sigma}}_x + \vec{\nabla} w_y \bullet \vec{\boldsymbol{\sigma}}_y\right) d\Omega = \int_{\Gamma_t} \vec{w} \bullet \vec{t}\, d\Gamma + \int_{\Omega} \vec{w} \bullet \vec{b}\, d\Omega \;\; \forall w_x \in U_0$$

$$\vec{\nabla} w_x \bullet \vec{\boldsymbol{\sigma}}_x + \vec{\nabla} w_y \bullet \vec{\boldsymbol{\sigma}}_y = \left(\nabla_S w\right)^T \boldsymbol{\sigma}$$

- The matrix form of the weak form becomes:

$$\int_{\Omega}\left(\nabla_S w\right)^T \boldsymbol{\sigma}\, d\Omega = \int_{\Gamma_t} \vec{w} \bullet \vec{t}\, d\Gamma + \int_{\Omega} \vec{w} \bullet \vec{b}\, d\Omega \;\; \forall w_x \in U_0$$

Weak form for isotropic linear elasticity

- Substituting the kinematic $\varepsilon = \nabla_S u$ and constitutive σ = **D**ε, relations, the resulting weak form in 2D can be summarized as:

Find $u \in U \; on \; \overline{\Omega} \; such \; that:$

$$\int_{\Omega} \left(\nabla_S w\right)^T \boldsymbol{\sigma} d\Omega = \int_{\Gamma_t} \vec{w} \bullet \vec{t} d\Gamma + \int_{\Omega} \vec{w} \bullet \vec{b} d\Omega \; \forall w \in U_0$$

$$where: U = \left\{ u : u \in H^1, u = \bar{u} \; on \; \Gamma_u \right\} \; and$$

$$U_0 = \left\{ w : w \in H^1, w = 0 \; on \; \Gamma_u \right\}$$

Can you think a virtual work interpretation of the above equation?