MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

Cornell University, Fall 2009

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Finite element approximation of 2D BVP

- Find a function $u(x,y) \in H^1(\Omega)$, such that $u(s) = \hat{u}(s),\ s \in \Gamma_u \equiv \partial\Omega_1$ and the following holds:

$$\int_{\Omega_1} \left[k(x,y)\vec{\nabla}u(x,y) \bullet \vec{\nabla}w(x,y) + b(x,y)u(x,y)w(x,y) \right] dxdy + \int_{\partial\Omega_2} puwds = \int_{\Omega} fwdxdy + \int_{\partial\Omega_2} \gamma wds$$

$$\textit{for all functions } w(x,y) \in H^1(\Omega) \textit{ with } w = 0 \textit{ on } \partial\Omega_1 \quad (\textit{here } \gamma = p\hat{u}).$$

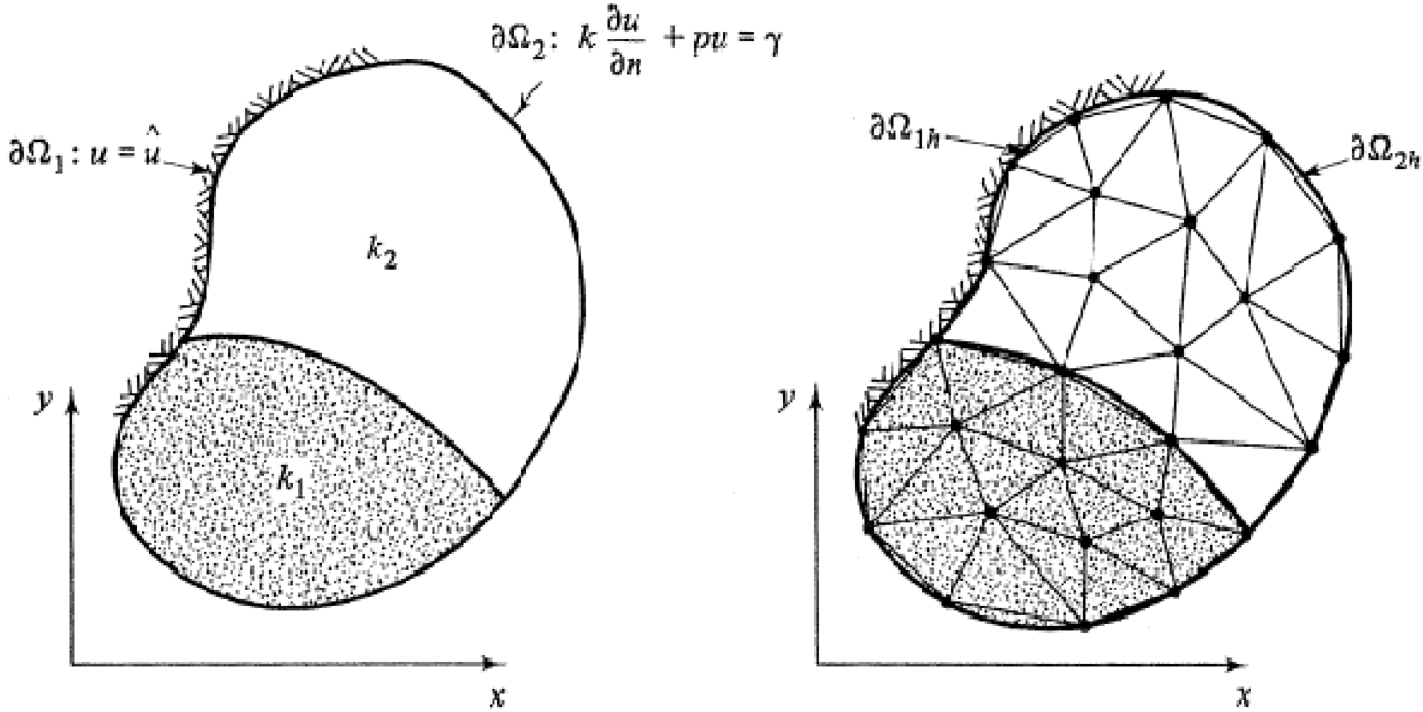
- Let us re-write this explicitly as:

$$\int_{\Omega_1} \left[k\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} \right) + b(x,y)u(x,y)w(x,y) \right] dxdy + \int_{\partial\Omega_2} puwds = \int_{\Omega} fwdxdy + \int_{\partial\Omega_2} \gamma wds$$

$$\textit{for all } w(x,y) \in H^1(\Omega) \textit{ with } w = 0 \textit{ on } \partial\Omega_1, \textit{where } \gamma = p\hat{u}.$$

Finite element approximation of 2D BVP

- We approximate the domain with E finite elements and N nodes placing nodes and elements in such a way that element boundaries coincide as close as possible to the interface with jump in k.



Finite element approximation of 2D BVP

- We define an N-dimensional subspace H^h of $H^1(\Omega_h)$ by constructing appropriate global basis functions $N_i, i = 1, 2, .., N$ using the elements we discussed earlier.
- A typical test function in H^h is of the form:

$$w_h(x,y) = \sum_{j=1}^{N} \underbrace{w_j}_{w_h(x_j, y_j)} N_j(x,y)$$

- In general the essential BC data (Dirichlet data) $\widehat{u} \; in \; \partial\Omega_1$ is approximated as:

$$\hat{u}_h(s) = \sum_j \hat{u}_j N_j(x(s), y(s))$$

where the sum is over all nodes on $\partial\Omega_{1h}$.

FEM problem statement

- Find a function $u_h \in H^h$, such that $u_h(x,y)=\sum_{j=1}^{N} u_j N_j(x,y)$ and $u_j = \hat{u}_j$ at the nodes on $\partial\Omega_{1h}$ so the following holds:

$$\int_{\Omega_h}\left[k\left(\frac{\partial u_h}{\partial x}\frac{\partial w_h}{\partial x}+\frac{\partial u_h}{\partial y}\frac{\partial w_h}{\partial y}\right)+b(x,y)u_h(x,y)w_h(x,y)\right]dxdy+\int_{\partial\Omega_{2h}} p u_h w_h ds=\int_{\Omega_h} f w_h dxdy+\int_{\partial\Omega_{2h}}\gamma w_h ds$$

$$for\ all\ w_h \in H^h\ with\ w_h = 0\ on\ \partial\Omega_{1h}.$$

- The above equation leads to the following system of algebraic equations:

$$\sum_{j=1}^{N} K_{ij}u_j = F_i, \quad i=1,2,..,N$$

$$K_{ij}=\int_{\Omega_h}\left[k\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)+b(x,y)N_iN_j\right]dxdy+\int_{\partial\Omega_{2h}} pN_iN_j ds,\ \ i,j=1,...,N$$

Symmetric, banded

$$F_i=\int_{\Omega_h} fN_i dxdy+\int_{\partial\Omega_{2h}}\gamma N_i ds$$

Finite element approximation

- Each of the integrals in the stiffness and load vector can be computed as the sum of contributions from each element in the mesh. But we need to approach this more carefully!
- Let Ω_e denote a typical finite element. The exact solution u on Ω_e of our BVP satisfies:

$$\int_{\Omega_e} \left[k \vec{\nabla} u \bullet \vec{\nabla} w + buw \right] dxdy = \int_{\Omega_e} f\, w dxdy - \int_{\partial\Omega_e} \sigma_n w ds$$

$$\textit{for all admissible functions } w(x,y),$$

$$\textit{where } \sigma_n \textit{ is the normal component of the flux at } \partial\Omega_e.$$

- Let u_h^e and w_h^e denote the restrictions of the approximations u_h and w_h to Ω_e. Then the local approximation of the variational BVP over Ω_e is:

$$\int_{\Omega_e} \left[k \vec{\nabla} u_h^e \bullet \vec{\nabla} w_h^e + b u_h^e w_h^e \right] dxdy = \int_{\Omega_e} f\, w_h^e dxdy - \int_{\partial\Omega_e} \underbrace{\sigma_n}_{\substack{\textit{exact flux on} \\ \partial\Omega_e (\textit{not known})}} w_h^e ds$$

Finite element approximation

$$\int_{\Omega_e} \left[k\vec{\nabla} u_h^e \bullet \vec{\nabla} w_h^e + b u_h^e w_h^e \right] dxdy = \int_{\Omega_e} f w_h^e dxdy - \int_{\partial\Omega_e} \underbrace{\sigma_n}_{\substack{exact\ flux\ on \\ \partial\Omega_e (not\ known)}} w_h^e ds$$

- Since $w_h = 0 \ on \ \partial\Omega_{1h}$, there will be no contribution in the last integral from elements with sides that coincide with $\partial\Omega_{1h}$.
- We already have in place the following approximations:

$$w_h^e(x) = \sum_{i=1}^{N_e} w_i^e N_i^e(x,y), \quad u_h^e(x) = \sum_{j=1}^{N_e} u_j^e N_j^e(x,y)$$

where $N_j^e(x,y)$ are the shape functions in Ω_e and N_e the number of nodes in Ω_e .

- The following linear system is then obtained:

Element equations

$$\sum_{j=1}^{N_e} k_{ij}^e u_j^e = f_i^e - \sigma_i^e, \quad i = 1, 2, .., N_e$$

$$k_{ij}^e = \int_{\Omega_e} \left[k \left(\frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} \right) + b N_i^e N_j^e \right] dxdy$$

Element stiffness matrix

$$f_i^e = \int_{\Omega_e} f N_i^e dxdy$$

Element load vector

$$\sigma_i^e = \int_{\partial\Omega_e} \sigma_n N_i^e ds$$

Element flux vector obtained by assigning to node *i* of Ω_e a weighted average of the actual flux σ_n across $\partial\Omega_e$.

Assembly process

- The global system of equations is obtained by summing over all elements *E* in the mesh.
  - We expand the element stiffness k^e to a matrix *NxN* K^e with zeros everywhere except those rows and columns corresponding to nodes within Ω_e and f^e and σ^e will be expanded to *Nx1* vectors F^e and Σ^e with nonzero entries only in those rows corresponding to nodes in Ω_e

$$\sum_{e=1}^{E}\int_{\Omega_e}\left[k\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)+bN_iN_j\right]dxdy=\sum_{e=1}^{E}K_{ij}^e,\ i,\ j=1,2,\ldots,N$$

$$\sum_{e=1}^{E}\int_{\Omega_e} fN_i dxdy=\sum_{e=1}^{E}F_i^e, i=1,2,\ldots,N$$

$$\sum_{e=1}^{E}\left(K_{ij}^e u_j - F_i^e + \Sigma_i^e\right)=0,\quad i=1,2,\ldots,N$$

Boundary conditions

$$\sum_{e=1}^{E}\left(K_{ij}^{e}u_{j}-F_{i}^{e}+\Sigma_{i}^{e}\right)=0,\quad i=1,2,\ldots,N$$

- Note that the contributions to K_{ij} and F_i from boundary conditions must enter the problem through the terms Σ_i^e .
  - We note that the sum of the contour integrals can be written as:

$$\sum_{e=1}^{E}\Sigma_{i}^{e}=S_{i}^{(0)}+S_{i}^{(1)}+S_{i}^{(2)},\quad i=1,2,\ldots,N$$

$$S_{i}^{(0)}=\sum_{e=1}^{E}\int_{\partial\Omega_{e}-\partial\Omega_{h}}\sigma_{n}N_{i}ds\qquad S_{i}^{(1)}=\sum_{e=1}^{E}\int_{\partial\Omega_{1h}}\sigma_{n}N_{i}ds\qquad S_{i}^{(2)}=\sum_{e=1}^{E}\int_{\partial\Omega_{2h}}\sigma_{n}N_{i}ds$$

$\partial\Omega_e-\partial\Omega_h$ Portion of the boundary $\partial\Omega_e$ of Ω_e not on $\partial\Omega_h$ (interelement boundaries)

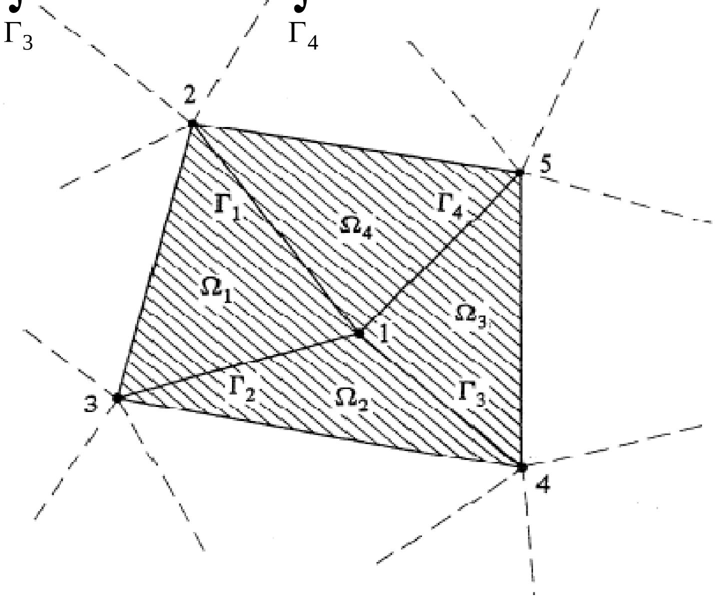
Boundary conditions

$$S_i^{(0)} = \sum_{e=1}^{E} \int_{\partial\Omega_e - \partial\Omega_h} \sigma_n N_i ds$$

- This vector is defined only at interior nodes. Consider the patch of 4 elements sharing node 1.

$$S_i^{(0)} = \sum_{e=1}^{E} \int_{\partial\Omega_e} \sigma_n N_1 ds = \int_{\Gamma_1} [\![\sigma_n]\!] N_1 ds + \int_{\Gamma_2} [\![\sigma_n]\!] N_1 ds \int_{\Gamma_3} [\![\sigma_n]\!] N_1 ds + \int_{\Gamma_4} [\![\sigma_n]\!] N_1 ds$$

- From the conservation law $[\![\sigma_n]\!] = 0$ across an interface where no point or line sources are applied. Thus if *f* is smooth in the patch shown $S_i^{(0)} = 0$



Boundary conditions

$$S_i^{(0)} = \sum_{e=1}^{E} \int_{\partial\Omega_e - \partial\Omega_h} \sigma_n N_i \, ds$$

- When the source *f* contains a line source or concentrated point source, then $[\![\sigma_n]\!]$ is equal to the intensity of the line source.
- We can include point sources by writing *f(x,y)* as

$$f(x,y) = \underbrace{\tilde{f}(x,y)}_{smooth\ part} + \underbrace{\hat{f}\,\delta(x - x_i, y - y_i)}_{point\ source\ at\ (x_i, y_i) \in \Omega_h}$$

- We assume that the mesh is constructed so that there is a node at the source location.

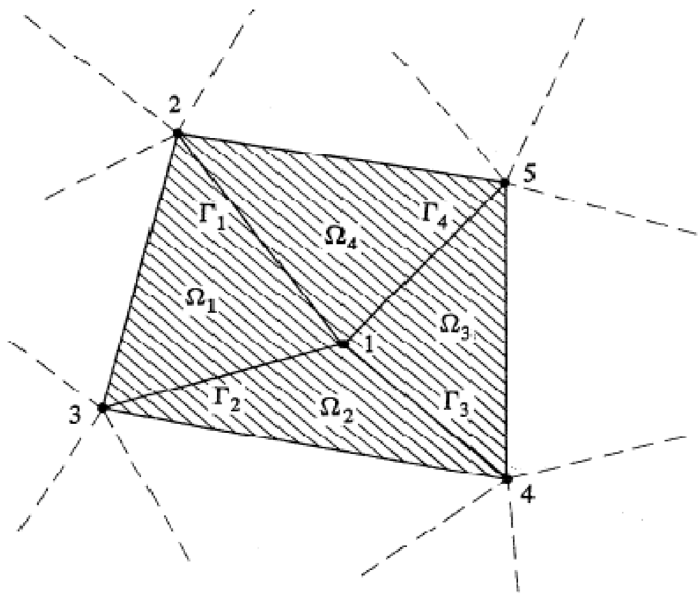
Boundary conditions

$$S_i^{(0)} = \sum_{e=1}^{E} \int_{\partial\Omega_e - \partial\Omega_h} \sigma_n N_i ds \qquad f(x,y) = \underbrace{\tilde{f}(x,y)}_{smooth\ part} + \underbrace{\hat{f}\delta(x-x_i, y-y_i)}_{point\ source\ at\ (x_i, y_i)\in\Omega_h}$$

- For source at node 1 in the figure, we have:

$$S_i^{(0)} = \sum_{e=1}^{4} \int_{\partial\Omega_e - \partial\Omega_h} \sigma_n N_i ds = \underbrace{\sum_{m=1}^{4} \int_{\Gamma_m} [\![\sigma_n]\!] N_1 ds}_{\text{weighted average of the jumps at node 1}}$$

$$S_i^{(0)} = \hat{f}$$



- The presence of sources leads to very singular solutions *u*.

Boundary conditions

$$S_i^{(1)} = \sum_{e=1}^{E} \int_{\partial\Omega_{1h}} \sigma_n N_i ds$$

- The values of u_h are prescribed at $\partial\Omega_{1h}$. Since σ_n is not known on $\partial\Omega_{1h}$, $S_i^{(1)}$ cannot be described here. However, once all nodal $u_1, u_2, \ldots u_N$ are computed, we can evaluate $S_i^{(1)}$ from

$$\sum_{e=1}^{E} \left(K_{ij}^e u_j - F_i^e + \Sigma_i^e \right) = 0, \quad i = 1, 2, ..., N$$

Boundary conditions

$$S_i^{(2)} = \sum_{e=1}^{E} \int_{\partial\Omega_{2h}} \sigma_n N_i ds$$

- On $\partial\Omega_{2h}$ the natural boundary condition is prescribed. There we set $\sigma_n(s) = p(s)u_h(s) - \gamma(s)$

$$S_i^{(2)} = \sum_{e=1}^{E} \int_{\partial\Omega_{2h}} \left[p \sum_{j=1}^{N} u_j N_j - \gamma \right] N_i ds = \sum_{j=1}^{N} P_{ij} u_j - \gamma_i$$

$$\gamma_i = \int_{\partial\Omega_{2h}} \gamma N_i ds = \sum_{e=1}^{E} \int_{\partial\Omega_{2h}^e} \gamma N_i ds = \sum_{e=1}^{E} \gamma_i^e$$

$$P_{ij} = \int_{\partial\Omega_{2h}} p N_i N_j ds = \sum_{e=1}^{E} \int_{\underbrace{\partial\Omega_{2h}^e}_{\substack{\text{portion of } \partial\Omega^e \\ \text{intersecting } \partial\Omega_{2h}}}} p N_i N_j ds = \sum_{e=1}^{E} P_{ij}^e$$

Final algebraic equations

- The final system of equations is given as:

$$\sum_{j=1}^{N} K_{ij} u_j = F_i - S_i^{(1)}, \quad i = 1, 2, \ldots, N$$

$$K_{ij} = \sum_{e=1}^{E} (K_{ij}^e + P_{ij}^e), \quad F_i = \sum_{e=1}^{E} (F_i^e + \gamma_i^e)$$

- Once boundary conditions are imposed on $\partial\Omega_{1h}$ we can proceed solving the system of equations for the unknown nodal values.

Example

- Consider the FEM solution of the following problem:

$$-\Delta u(x,y) = f(x,y) \; in \; \Omega$$

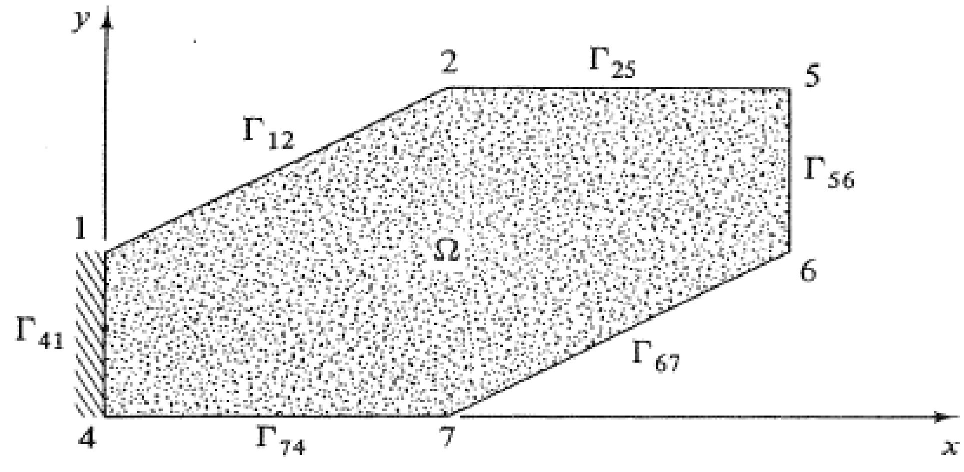
$$u = 0 \; on \; \Gamma_{41}$$

$$\frac{\partial u}{\partial n} = 0 \; on \; \Gamma_{12}, \Gamma_{25,} \Gamma_{67}, \Gamma_{74,}$$

$$\frac{\partial u}{\partial n} + \beta u = \gamma \; on \; \Gamma_{56}$$

$$\partial\Omega_{1h} = \partial\Omega_1$$

$$\partial\Omega_{2h} = \partial\Omega_2$$



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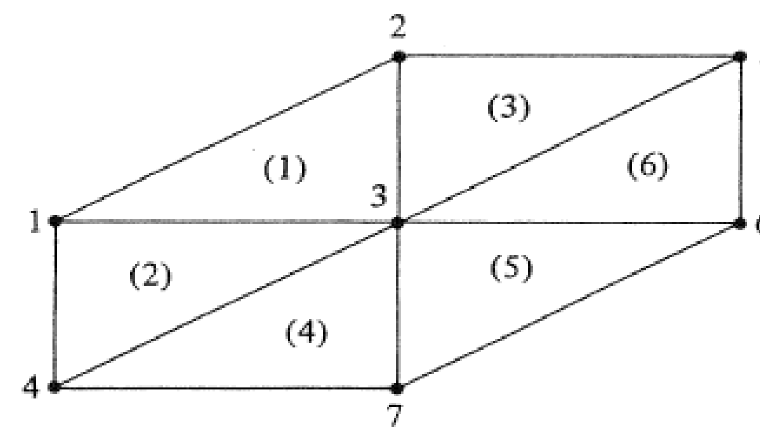


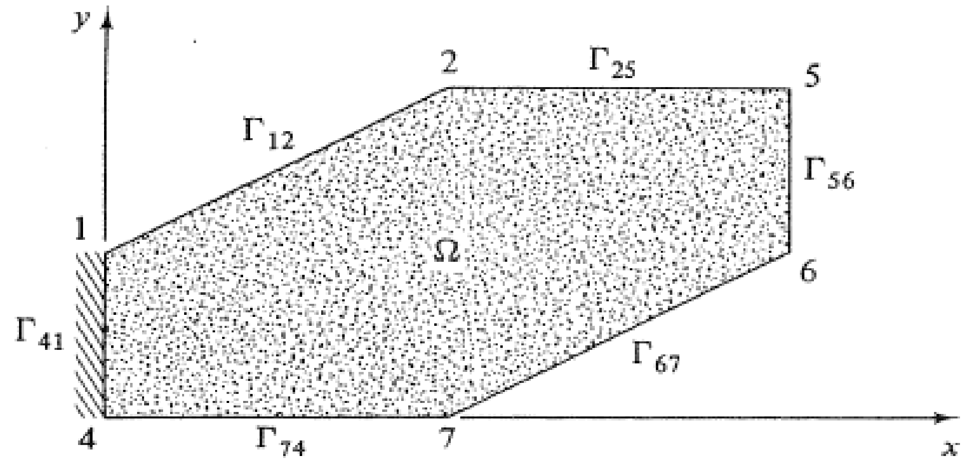
$$\partial\Omega_2 = \Gamma_{12} \cup \Gamma_{25} \cup \Gamma_{56} \cup \Gamma_{67} \cup \Gamma_{74},$$

$$\partial\Omega_1 = \Gamma_{41}$$

Example

- Element 1:

$$K^1 = \begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & 0 & 0 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

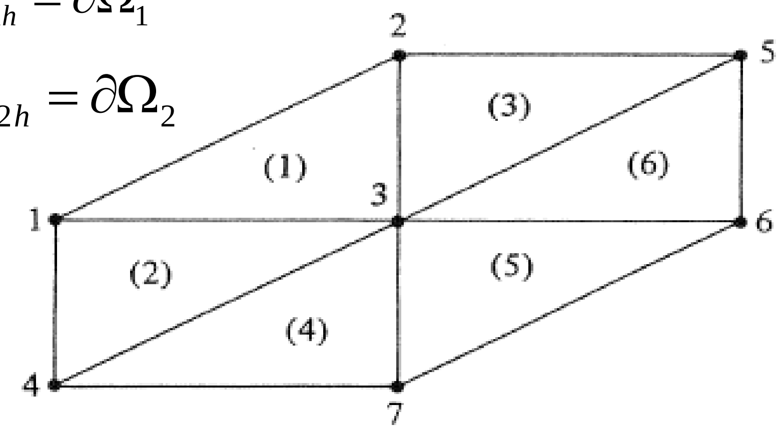


(a)

$$\partial\Omega_{1h} = \partial\Omega_1$$

$$\partial\Omega_{2h} = \partial\Omega_2$$

$$F^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\partial\Omega_2 = \Gamma_{12} \cup \Gamma_{25} \cup \Gamma_{56} \cup \Gamma_{67} \cup \Gamma_{74},$$

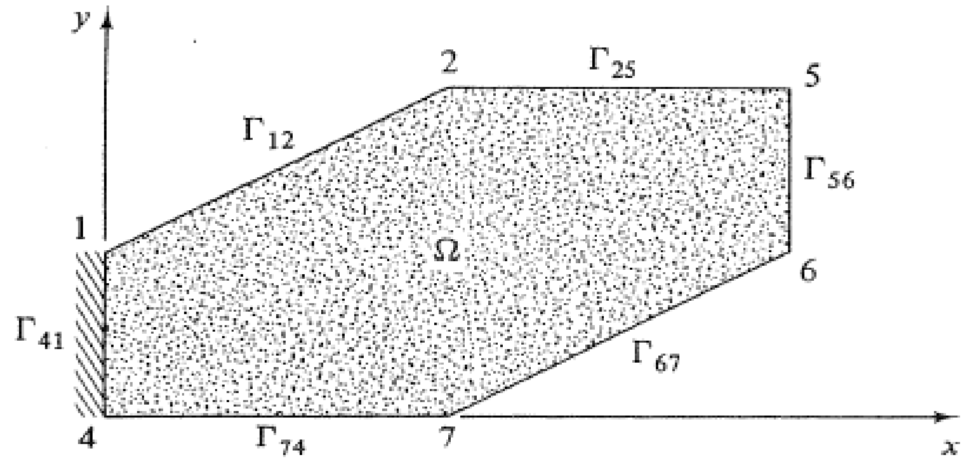
$$\partial\Omega_1 = \Gamma_{41}$$

Example

- Element 2:

$$K^2 = \begin{bmatrix} k_{11}^2 & 0 & k_{12}^2 & k_{13}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{21}^2 & 0 & k_{22}^2 & k_{23}^2 & 0 & 0 & 0 \\ k_{31}^2 & 0 & k_{32}^2 & k_{33}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F^2 = \begin{bmatrix} f_1^2 \\ 0 \\ f_2^2 \\ f_3^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



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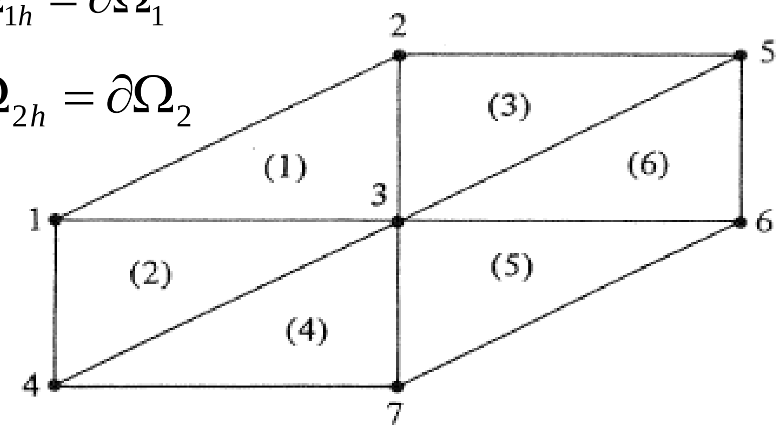
$$\partial\Omega_{1h} = \partial\Omega_1$$

$$\partial\Omega_{2h} = \partial\Omega_2$$



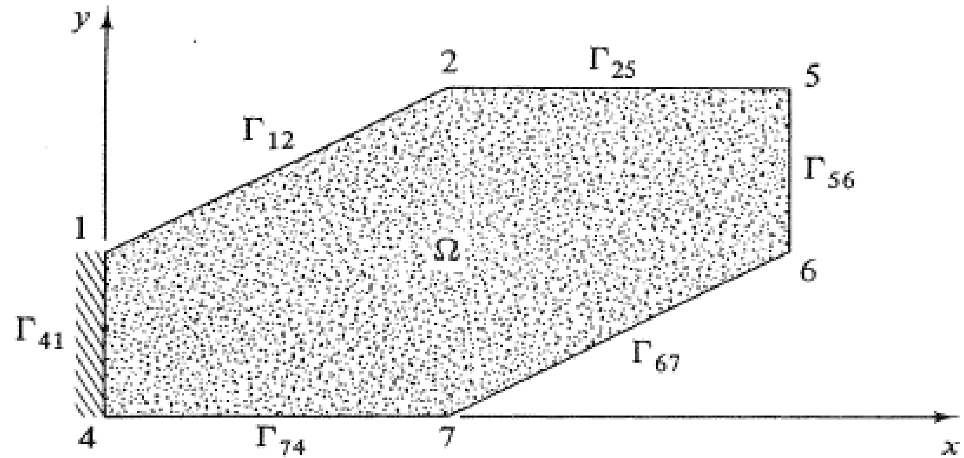
$$\partial\Omega_2 = \Gamma_{12} \cup \Gamma_{25} \cup \Gamma_{56} \cup \Gamma_{67} \cup \Gamma_{74},$$

$$\partial\Omega_1 = \Gamma_{41}$$

Example

- Element 6:

$$K^6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^6 & 0 & k_{12}^6 & k_{13}^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{21}^6 & 0 & k_{22}^6 & k_{23}^6 & 0 \\ 0 & 0 & k_{31}^6 & 0 & k_{32}^6 & k_{33}^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



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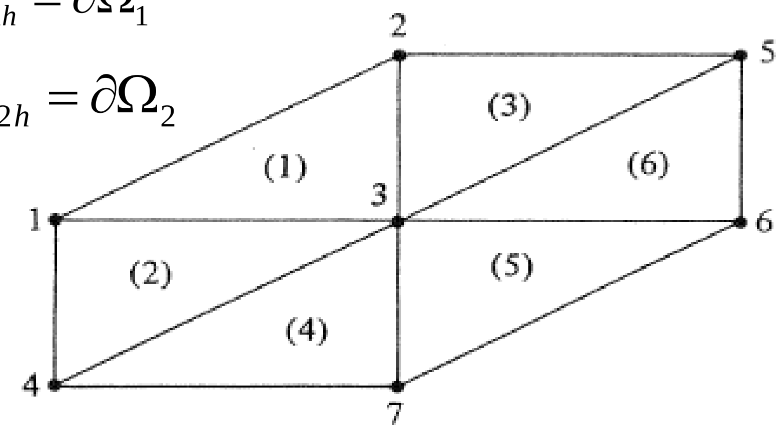
$$\partial\Omega_{1h} = \partial\Omega_1$$

$$\partial\Omega_{2h} = \partial\Omega_2$$



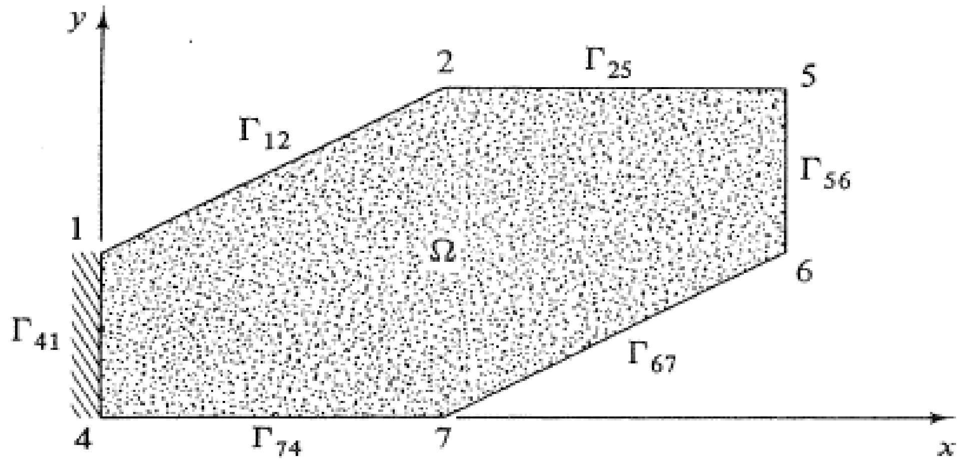
$$F^1 = \begin{bmatrix} 0 \\ 0 \\ f_1^6 \\ 0 \\ f_2^6 \\ f_3^6 \\ 0 \end{bmatrix}$$

$$\partial\Omega_2 = \Gamma_{12} \cup \Gamma_{25} \cup \Gamma_{56} \cup \Gamma_{67} \cup \Gamma_{74},$$

$$\partial\Omega_1 = \Gamma_{41}$$

Example problem: Assembly

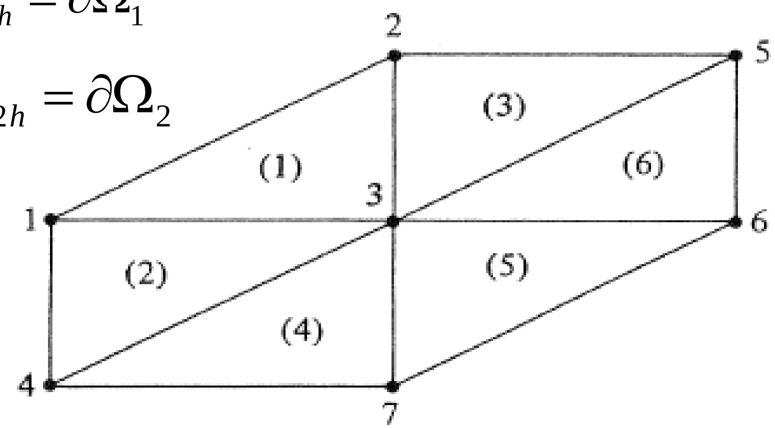
$$
K = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & 0 & 0 & 0 \\
K_{21} & K_{22} & K_{23} & 0 & K_{25} & 0 & 0 \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\
K_{41} & 0 & K_{43} & K_{44} & 0 & 0 & K_{47} \\
0 & K_{52} & K_{53} & 0 & \widetilde{K}_{55} & \widetilde{K}_{56} & 0 \\
0 & 0 & K_{63} & 0 & \widetilde{K}_{65} & \widetilde{K}_{66} & K_{67} \\
0 & 0 & K_{73} & K_{74} & 0 & K_{76} & K_{77}
\end{bmatrix}
$$



(a)

$$\partial\Omega_{1h} = \partial\Omega_1$$

$$\partial\Omega_{2h} = \partial\Omega_2$$

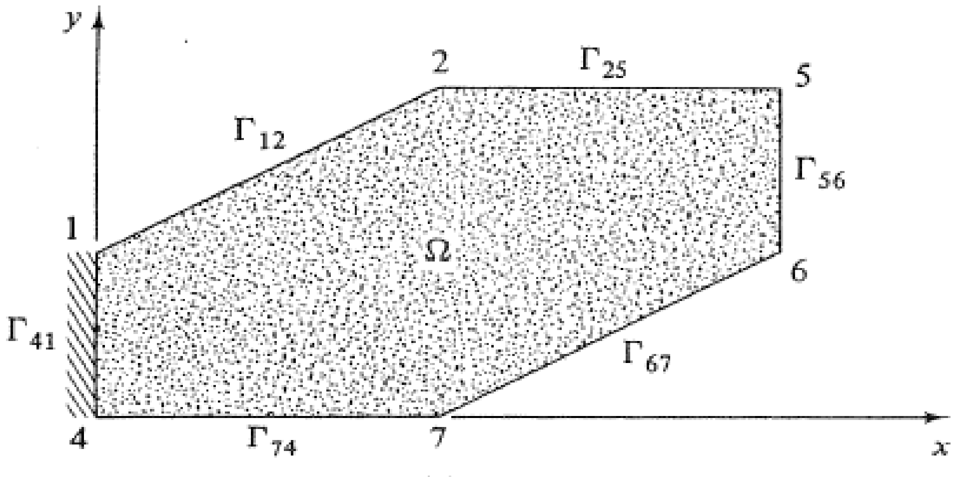


$$
F^1 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \widetilde{F}_5 \\ \widetilde{F}_6 \\ F_7 \end{bmatrix} - \begin{bmatrix} \Sigma_1 \\ 0 \\ 0 \\ \Sigma_4 \\ \Sigma_5 \\ \Sigma_6 \\ 0 \end{bmatrix}
$$

The stiffness terms with the symbol $\widetilde{K}$ will be modified once natural BC are applied.

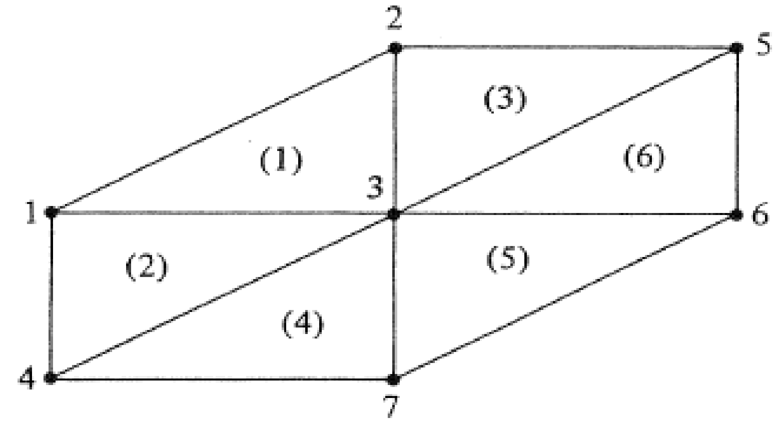
Natural boundary conditions

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{55} & P_{56} & 0 \\ 0 & 0 & 0 & 0 & P_{65} & P_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



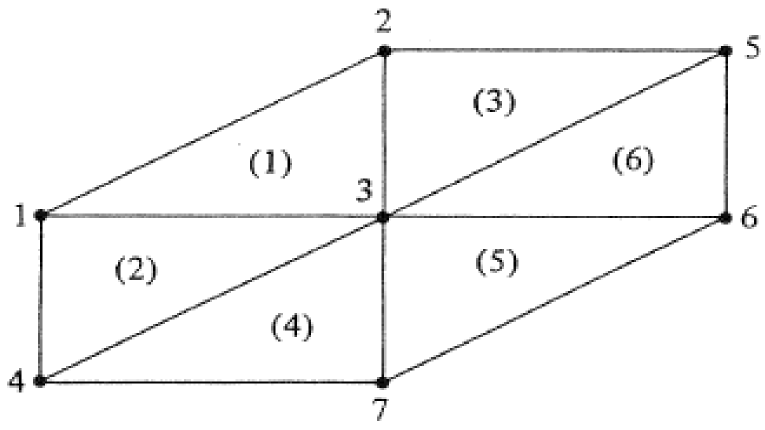
(a)

$$\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma_5 \\ \gamma_6 \\ 0 \end{bmatrix}$$



Final system of equations

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & K_{25} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\ K_{41} & 0 & K_{43} & K_{44} & 0 & 0 & K_{47} \\ 0 & K_{52} & K_{53} & 0 & K_{55} & K_{56} & 0 \\ 0 & 0 & K_{63} & 0 & K_{65} & K_{66} & K_{67} \\ 0 & 0 & K_{73} & K_{74} & 0 & K_{76} & K_{77} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} = \begin{Bmatrix} F_1 - \Sigma_1 \\ F_2 \\ F_3 \\ F_4 - \Sigma_4 \\ F_5 \\ F_6 \\ F_7 \end{Bmatrix}$$



Impose essential boundary conditions

$$\begin{bmatrix} K_{22} & K_{23} & K_{25} & 0 & 0 \\ K_{32} & K_{33} & K_{35} & K_{36} & K_{37} \\ K_{52} & K_{53} & K_{55} & K_{56} & 0 \\ 0 & K_{63} & K_{65} & K_{66} & K_{67} \\ 0 & K_{73} & 0 & K_{76} & K_{77} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_5 \\ F_6 \\ F_7 \end{Bmatrix}$$

Post-processing

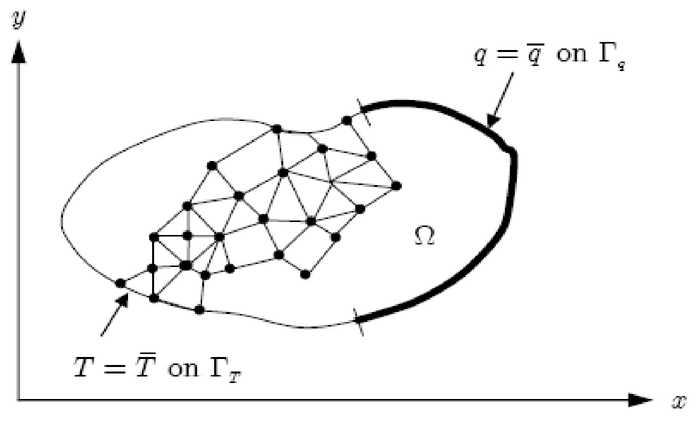
$$-\Sigma_1 = K_{12}u_2 + K_{13}u_3 + K_{14}u_4 - F_1$$

$$-\Sigma_4 = K_{43}u_3 + K_{47}u_7 - F_4$$

Revisiting the FEM formulation for 2D scalar field problems

- In the earlier part of this lecture, we used the stiffness matrix and load vectors in component forms.
- We will next discuss a similar example (notation is used from heat conduction) repeating the same calculations but in a matrix form. These calculations have been programmed in the 2dBVP MatLab software.
- Let us consider the following 2D BVP: Compute *T(x,y)*:

$$-\nabla \bullet (D \nabla T) = f(x,y)$$
$$T = \overline{T} \; on \; \Gamma_T$$
$$q = -D \nabla T \bullet n = \overline{q} \; on \; \Gamma_q$$



2D Heat conduction – weak form

- The weak statement for this problem takes the form:

Find a function $T(x,y) \in H^1(\Omega)$, with $T(s) = \overline{T}(s),\ s \in \Gamma_T$ such that:

$$\int_\Omega \{\nabla w\}^T [D]\{\nabla T\}\, d\Omega = \int_\Omega w^T f\, d\Omega - \int_{\Gamma_q} w^T \overline{q}\, d\Gamma$$

$$for\ all\ w(x,y) \in U_0$$

- We denote here $U_0 : w \in H^1(\Omega)\ with\ w = 0\ on\ \partial\Gamma_T$
- The weak form is written in a matrix form ready for FEM discretization, e.g.

$$\{\nabla T\} = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}, \{\nabla w\}^T = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}, \underbrace{q}_{\substack{heat\\flux}} = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = -[D]\{\nabla T\}, \underbrace{[D]}_{\substack{Conductivity\\matrix}} = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}$$

2D Heat conduction – weak form

- We replace the integrals with sum of integrals over the finite elements:

Find a function $T(x,y) \in H^1(\Omega)$, with $T(s) = \overline{T}(s),\ s \in \Gamma_T$ such that:

$$\sum_{e=1}^{nel} \int_{\Omega^e} \left\{\nabla w^e\right\}^T [D^e] \left\{\nabla T^e\right\} d\Omega = \sum_{e=1}^{nel} \int_{\Omega^e} w^{e^T} f\, d\Omega - \sum_{e=1}^{nel} \int_{\Gamma_q} w^{e^T} \overline{q} d\Gamma$$

$$for\ all\ w(x,y) \in U_0$$

- The finite element interpolation formula for the temperature in each element is:

$$T^e(x,y) = N^e(x,y) d^e = \underbrace{\begin{bmatrix} N_1^e & N_2^e & \dots & N_{nen}^e \end{bmatrix}}_{[N]\ matrix} \underbrace{\begin{bmatrix} T_1^e \\ T_2^e \\ \dots \\ T_{nen}^e \end{bmatrix}}_{\{d^e\}} = \left[N^e\right]\left\{d^e\right\} = \left[N^e\right] \underbrace{\left[L^e\right]}_{\substack{scatter \\ matrix}} \underbrace{\{d\}}_{\substack{global \\ nodal \\ temperatures}}$$

2D Heat conduction – FE interpolation

- The gradient fields of T^e and w^e are obtained as:

$$\left\{\nabla T^e\right\} = \begin{bmatrix} \frac{\partial T^e}{\partial x} \\ \frac{\partial T^e}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} T_1^e + \frac{\partial N_2^e}{\partial x} T_2^e + \ldots + \frac{\partial N_{nen}^e}{\partial x} T_{nen}^e \\ \frac{\partial N_1^e}{\partial y} T_1^e + \frac{\partial N_2^e}{\partial y} T_2^e + \ldots + \frac{\partial N_{nen}^e}{\partial y} T_{nen}^e \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \ldots & \frac{\partial N_{nen}^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \ldots & \frac{\partial N_{nen}^e}{\partial y} \end{bmatrix}}_{\text{The } \left[B^e\right] \text{ matrix}} \underbrace{\begin{bmatrix} T_1^e \\ T_2^e \\ \ldots \\ T_{nen}^e \end{bmatrix}}_{\left\{d^e\right\}} = \left[B^e(x,y)\right]\left\{d^e\right\} = \left[B^e\right]\underbrace{\left[L^e\right]}_{\substack{\text{scatter}\\ \text{matrix}}}\underbrace{\left\{d\right\}}_{\substack{\text{global}\\ \text{nodal}\\ \text{temperatures}}}$$

Here, nen: number element nodes

- Similarly for the gradient of w^e:

$$\left\{\nabla w^e\right\}^T = \begin{bmatrix} \frac{\partial w^e}{\partial x} & \frac{\partial w^e}{\partial y} \end{bmatrix} = \left\{w^e\right\}^T \left[B^e(x,y)\right]^T = \underbrace{\begin{bmatrix} w_1^e & w_2^e & .. & w_{nen}^e \end{bmatrix}}_{\left\{w^e\right\}^T} \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_1^e}{\partial y} \\ \frac{\partial N_2^e}{\partial x} & \frac{\partial N_2^e}{\partial y} \\ \ldots & \ldots \\ \frac{\partial N_{nen}^e}{\partial x} & \frac{\partial N_{nen}^e}{\partial y} \end{bmatrix} = \left\{w\right\}^T \underbrace{\left[L^e\right]^T}_{\substack{\text{scatter}\\ \text{matrix}}} \left[B^e\right]^T$$

2D Heat conduction – weak form

- We can substitute these expressions into the weak form

$$\sum_{e=1}^{nel}\int_{\Omega^e}\left\{\nabla w^e\right\}^T [D^e]\left\{\nabla T^e\right\} d\Omega = \sum_{e=1}^{nel}\int_{\Omega^e} w^{e^T} f\, d\Omega - \sum_{e=1}^{nel}\int_{\Gamma_q^e} w^{e^T}\bar{q}\, d\Gamma \Rightarrow$$

$$\{w\}^T \sum_{e=1}^{nel}\left[L^e\right]^T\left[\int_{\Omega^e}\left[B^e\right]^T [D^e]\left[B^e\right] d\Omega\left[L^e\right]\{d\} - \int_{\Omega^e}\left[N^e\right]^T f\, d\Omega + \int_{\Gamma_q^e}\left[N^e\right]^T \bar{q}\, d\Gamma d\Omega\right] = 0 \;\forall \{w_F\}$$

- Note as we did earlier for the 1D BVP, the vector *w* was partitioned as follows:

$$\{d\} = \begin{bmatrix}\bar{d}_E \\ d_F\end{bmatrix}, \{w\} = \begin{bmatrix} w_E = 0 \\ w_F\end{bmatrix} = \begin{bmatrix} 0 \\ w_F\end{bmatrix}$$

2D Heat conduction – weak form

$$\{w\}^T \sum_{e=1}^{nel} \left[L^e\right]^T \left[\int_{\Omega^e} \left[B^e\right]^T [D^e] \left[B^e\right] d\Omega \left[L^e\right] \{d\} - \int_{\Omega^e} \left[N^e\right]^T f\, d\Omega + \int_{\Gamma_q^e} \left[N^e\right]^T \bar{q}\, d\Gamma \right] = 0 \;\forall \{w_F\}$$

- From this equation, we can easily identify:

$$\left[K^e\right] = \int_{\Omega^e} \left[B^e\right]^T [D^e] \left[B^e\right] d\Omega \qquad \left\{f^e\right\} = \underbrace{\int_{\Omega^e} \left[N^e\right]^T f\, d\Omega}_{\left\{f_\Omega^e\right\}} - \underbrace{\int_{\Gamma_q^e} \left[N^e\right]^T \bar{q}\, d\Gamma}_{\left\{f_\Gamma^e\right\}}$$

- We now can write the weak form as:

$$\{w\}^T \left[\underbrace{\left(\sum_{e=1}^{nel} \left[L^e\right]^T \left[K^e\right] \left[L^e\right] \right) \{d\} - \left(\sum_{e=1}^{nel} \left[L^e\right]^T f^e \right)}_{\text{Residual } \{r\}} \right] = 0 \;\forall \{w_F\}$$

2D Heat conduction – global stiffness and load

$$\{w\}^T \left[\underbrace{\left(\sum_{e=1}^{nel} \left[L^e\right]^T \left[K^e\right]\left[L^e\right] \right)\{d\} - \left(\sum_{e=1}^{nel} \left[L^e\right]^T f^e \right)}_{\text{Residual } \{r\}} \right] = 0 \; \forall \{w_F\}$$

- The global stiffness and load can now be written:

$$[K] = \sum_{e=1}^{nel} \left[L^e\right]^T \left[K^e\right]\left[L^e\right] \quad \{f\} = \sum_{e=1}^{nel} \left[L^e\right]^T \{f^e\}$$

- Utilizing the partition of *w*, the weak form is written as:

$$\{w_F\}^T \{r_F\} + \{w_E\}^T \{r_E\} = 0, \forall \{w_F\} \Rightarrow \{r_F\} = 0 \Rightarrow \{r\} = \begin{bmatrix} r_E \\ r_F = 0 \end{bmatrix} = \begin{bmatrix} K_E & K_{EF} \\ K_{FE} & K_F \end{bmatrix} \begin{bmatrix} \overline{d}_E \\ d_F \end{bmatrix} - \begin{bmatrix} f_E \\ f_F \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} K_E & K_{EF} \\ K_{FE} & K_F \end{bmatrix} \begin{bmatrix} \overline{d}_E \\ d_F \end{bmatrix} = \begin{bmatrix} f_E + r_E \\ f_F \end{bmatrix}$$

This follows exactly the solver in our MatLab 2DBVP program. r_E here are the fluxes on the boundaries with essential BCs. They are computed after this system of equations is solved for d_F.