MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

Cornell University, Fall 2009

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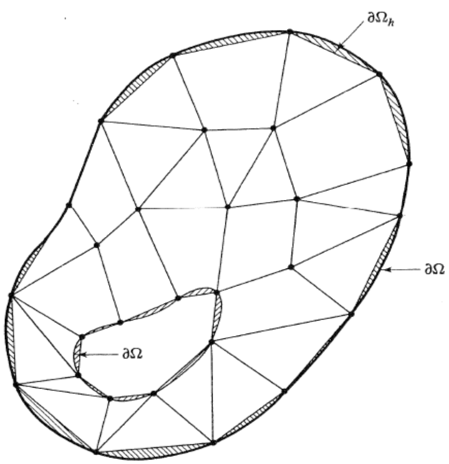
2D finite element discretization

- The basic idea is to represent the approximate function v_h (e.g. approximate solution u_h and test/weight functions w_h) by polynomials defined piecewise over geometrically simple (triangles, quadrilaterals, etc) subdomains of the 2D domain Ω^h.
- If the domain Ω is curved, we will always have discretization error since Ω^h will not perfectly match the given domain Ω. However, with mesh refinement, $\Omega_h \rightarrow \Omega.$



Two-dimensional discretization

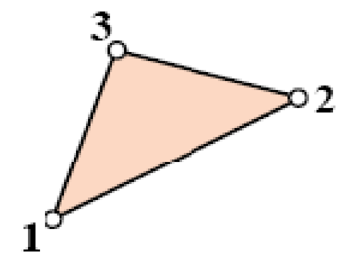
- There is a natural correspondence between the number and location of nodal points in an element and the number of terms used in the local polynomial approximation.
  - Recall that in 1D for a linear 2-node element, $v_h^e(x) = a_1 + a_2 x$. Since the element has 2 nodes, the 2 constants a_1, a_2 can uniquely be specified from the nodal values $v_h^e(x_1), v_h^e(x_2)$.
  - With this approach by requiring the functions $v_h^e(x)$ *and* $v_h^{e+1}(x)$, in adjacent elements *e* and *e+1* to share values at the same node, we produce a continuous function $v_h(x)$.

Two-dimensional discretization

- What polynomial approximations can we use in 2D?

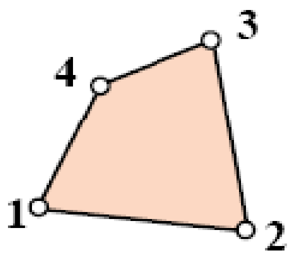
$$v_h^e(x) = a_1 + a_2 x + a_3 y$$

A triangle with 3 nodes at the vertices



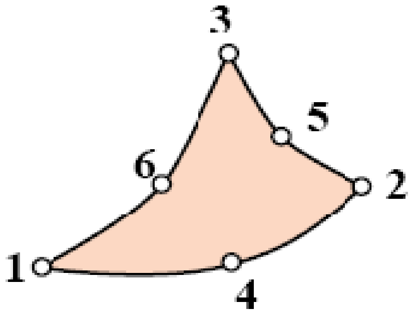
$$v_h^e(x) = a_1 + a_2 x + a_3 y + a_4 xy$$

A rectangle with 4 nodes at the vertices



$$v_h^e(x) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

A triangle with a node at each vertex and a node at the middle of each side!



Global FE basis functions

- Our goal is to construct an interpolant $f_h(x)$ of a function $f(x)$ in the domain Ω of the following form:

$$f_h(x,y)=\sum_{i=1}^{N} f_i N_i(x,y),\ N = number\ of\ nodes \text{ in } \Omega_h$$

- The basis functions $N_i(x,y)$ are defined such that:

$$N_i(x_j,y_j)=\begin{Bmatrix} 1\ if\ \ i=j \\ 0\ if\ \ i\neq j \end{Bmatrix}=\delta_{ij}\ (Kronecker\ delta)$$

where (x_j,y_j) are the coordinates of the finite element nodes in the mesh.

- Using this definition of the basis functions, note that

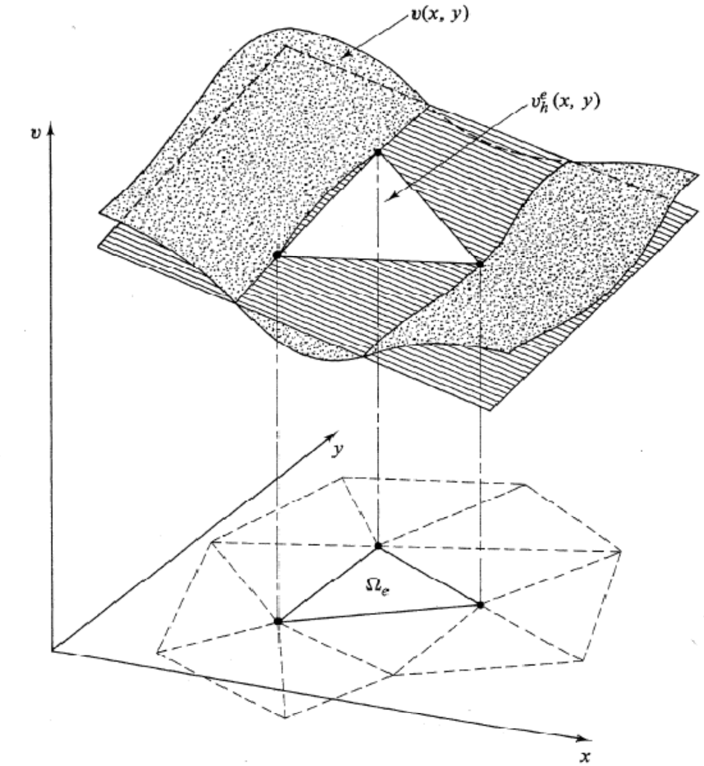
$$f_h(x_j,y_j)=\sum_{i=1}^{N} f_i N_i(x_j,y_j)=\sum_{i=1}^{N} f_i\delta_{ij}=f_j$$

Piecewise-linear interpolation on triangles

- Consider that Ω^h consists of E triangular elements and that we have linear interpolation on each element.

$$v_h^e(x) = a_1 + a_2 x + a_3 y$$

- The three values of $v_h^e(x)$ at the vertices of Ω_e determine a plane which intercepts the surface *v(x,y)* at 3 points.



Piecewise-linear interpolation on triangles

$$v_h^e(x) = a_1 + a_2 x + a_3 y$$

- We determine the 3 constants as follows:

$$v_1 \equiv v_h^e(x_1, y_1) = a_1 + a_2 x_1 + a_3 y_1,$$
$$v_2 \equiv v_h^e(x_2, y_2) = a_1 + a_2 x_2 + a_3 y_2,$$
$$v_3 \equiv v_h^e(x_3, y_3) = a_1 + a_2 x_3 + a_3 y_3,$$

- Solving this system of equations leads to:

$$a_1 = \frac{1}{2A_e}\left[v_1(x_2 y_3 - x_3 y_2) + v_2(x_3 y_1 - x_1 y_3) + v_3(x_1 y_2 - x_2 y_1)\right]$$

$$a_2 = \frac{1}{2A_e}\left[v_1(y_2 - y_3) + v_2(y_3 - y_1) + v_3(y_1 - y_2)\right]$$

$$a_3 = \frac{1}{2A_e}\left[v_1(x_3 - x_2) + v_2(x_1 - x_3) + v_3(x_2 - x_1)\right]$$

$$A_e = area\ of\ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

Piecewise-linear interpolation on triangles

$$v_h^e(x) = a_1 + a_2 x + a_3 y$$

$$a_1 = \frac{1}{2A_e}\left[v_1(x_2y_3 - x_3y_2) + v_2(x_3y_1 - x_1y_3) + v_3(x_1y_2 - x_2y_1)\right]$$

$$a_2 = \frac{1}{2A_e}\left[v_1(y_2 - y_3) + v_2(y_3 - y_1) + v_3(y_1 - y_2)\right]$$

$$a_3 = \frac{1}{2A_e}\left[v_1(x_3 - x_2) + v_2(x_1 - x_3) + v_3(x_2 - x_1)\right]$$

- Substituting the coefficients in the first approximation:

$$v_h^e(x) = v_1 N_1^e(x,y) + v_2 N_2^e(x,y) + v_3 N_3^e(x,y)$$

where:

$$N_1^e(x,y) = \frac{1}{2A_e}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$N_2^e(x,y) = \frac{1}{2A_e}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$N_3^e(x,y) = \frac{1}{2A_e}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

Linear shape functions over a triangular element

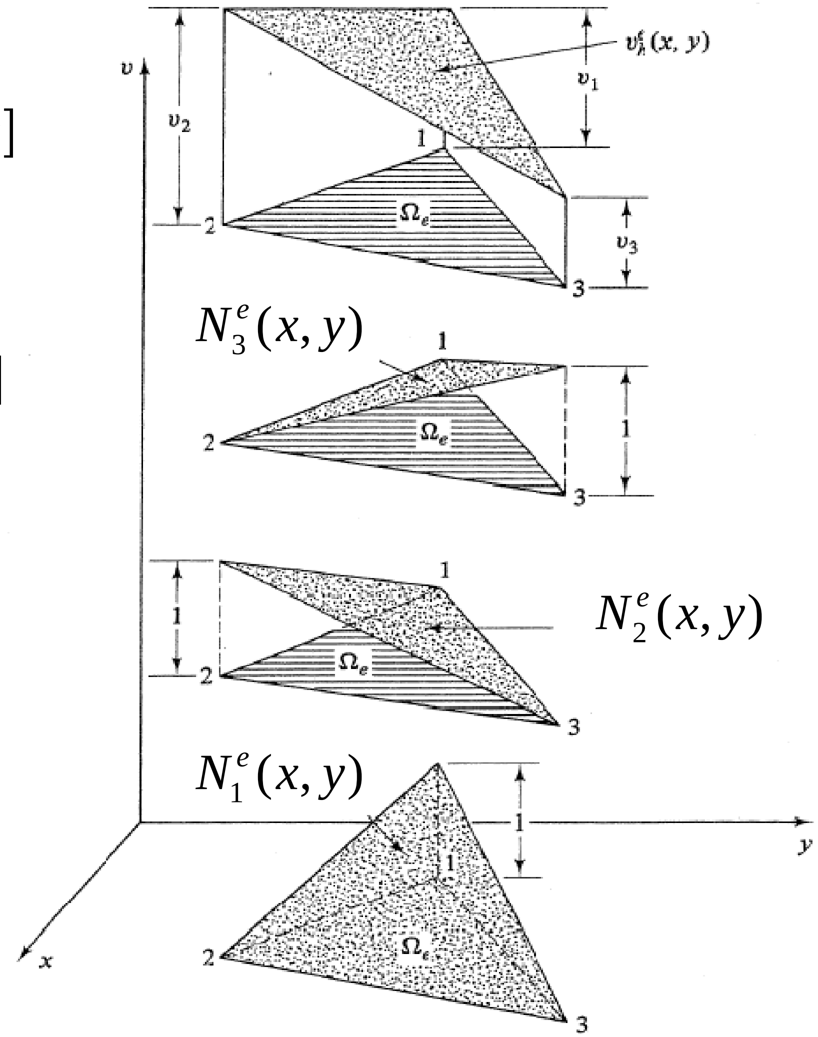
$$N_1^e(x,y) = \frac{1}{2A_e}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$N_2^e(x,y) = \frac{1}{2A_e}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$N_3^e(x,y) = \frac{1}{2A_e}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

- Please note that:

$$N_i^e(x_j, y_j) = \delta_{ij}$$



$$v_h^e(x) = v_1 N_1^e(x,y) + v_2 N_2^e(x,y) + v_3 N_3^e(x,y)$$

Linear shape functions over a 3-node triangular element

- The shape functions for an element e can be summarized in a `matrix form' as:

$$N^e = \begin{bmatrix} N_1^e & N_2^e & N_3^e \end{bmatrix}$$

$$N_1^e = \frac{1}{2A^e}(x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y)$$

$$N_2^e = \frac{1}{2A^e}(x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y)$$

$$N_3^e = \frac{1}{2A^e}(x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y)$$

$$2A^e = \left(x_2^e y_3^e - x_3^e y_2^e\right) - \left(x_1^e y_3^e - x_3^e y_1^e\right) + \left(x_1^e y_2^e - x_2^e y_1^e\right)$$

- Thus our interpolation formula in matrix form is:

$$v_h^e(x) = [N^e]\begin{bmatrix} v_1^e \\ v_2^e \\ v_3^e \end{bmatrix}$$

Linear shape functions over a 3-node triangular element

- Differentiation of the interpolation formula produces the matrix B^e that relates the gradient of $v_h^e(x)$ with the nodal values v_1^e, v_2^e, v_3^e.

$$\begin{bmatrix} \dfrac{\partial v^e}{\partial x} \\ \dfrac{\partial v^e}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{bmatrix}}_{\left[B^e\right]} \begin{bmatrix} v_1^e \\ v_2^e \\ v_3^e \end{bmatrix}$$

- Note that the matrix B^e is constant for 3 node triangulars (constant `strain triangulars')

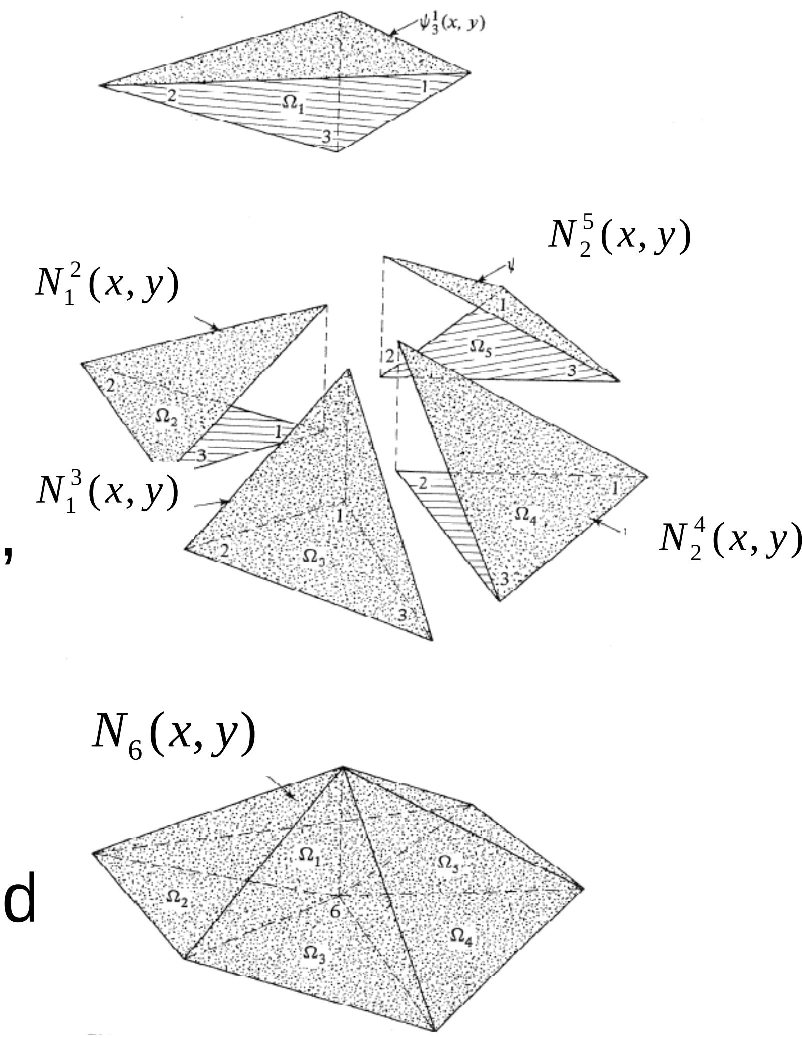
$$\underbrace{\left[B^e\right]}_{\substack{constant \\ matrix}} = \begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{bmatrix} = \frac{1}{2A^e} \begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^1 & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix}$$

Patching together linear basis functions

- What are the global basis functions $N_i, i = 1, 2, .., N$ that N_i^e produce?
- The basis functions N_i^e of adjacent elements are patched together to produce a pyramid function N_i at each global node i, such that:

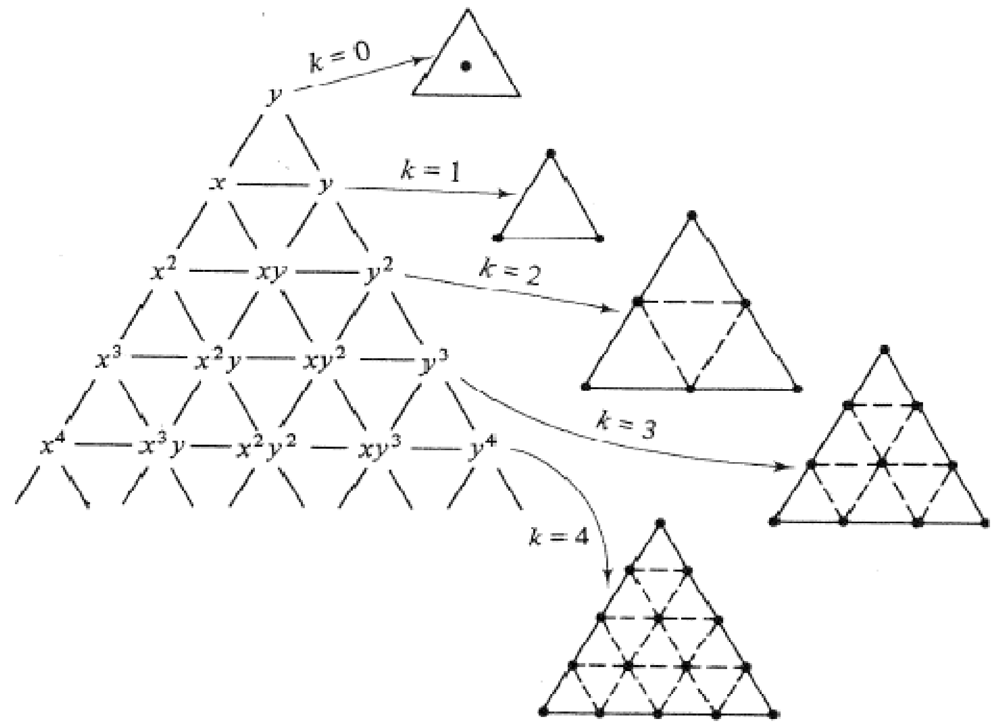
$$N_i(x_j, y_j) = \delta_{ij}$$

- For boundary nodes, the basis function is a portion of a pyramid



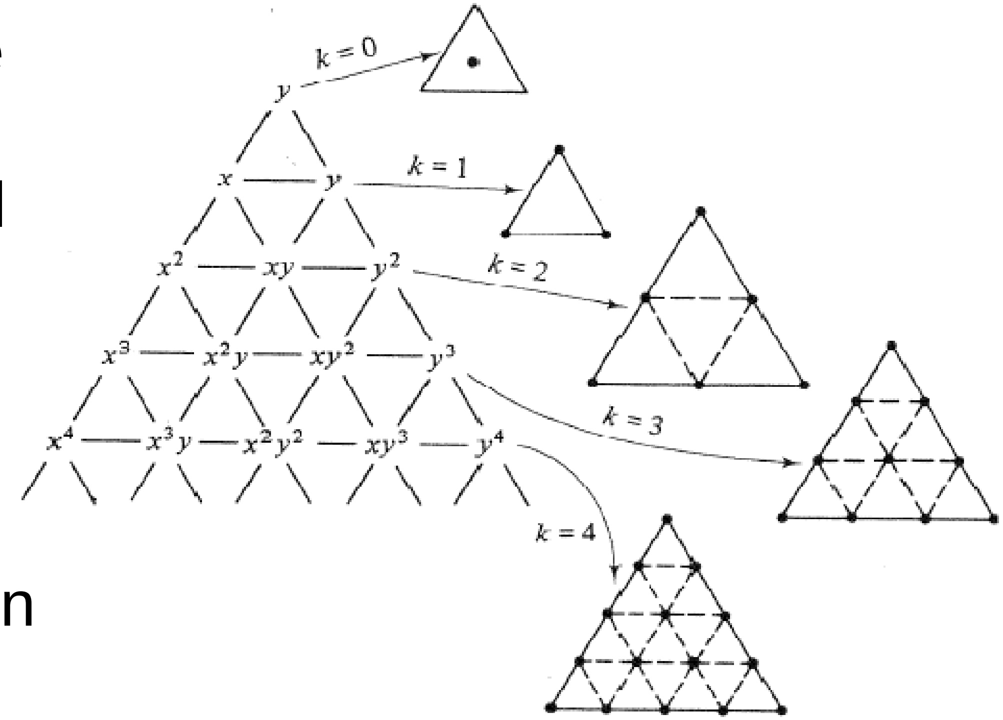
Pascal's triangle: Higher order triangle elements

- Triangular elements using higher degree polynomials can be easily constructed using the so called Pascal's triangle:
- A complete polynomial of degree *k* in *x* and *y* will have exactly $\frac{1}{2}(k+1)(k+2)$ terms.
- Thus a polynomial of degree *k* can be determined uniquely by specifying its value at $\frac{1}{2}(k+1)(k+2)$ points on the plane.



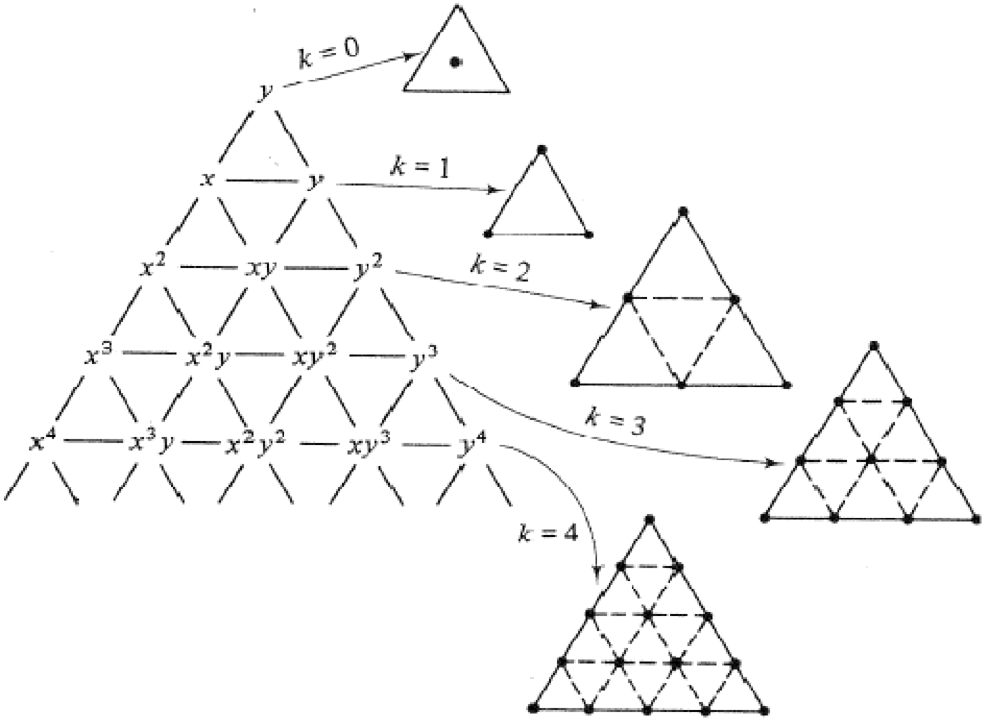
Other triangular elements

- Pascal's triangle implies a symmetric location of nodal points in triangular elements.
- For example, the 6 terms in a quadratic polynomial are determined by specifying the value of $v_h^e(x)$ at 6 nodal points, one at each vertex and one at each midpoint of each side.
- Note that this is precisely the location of the entries in the triangle formed by the quadratic in Pascal's triangle!



Other triangular elements

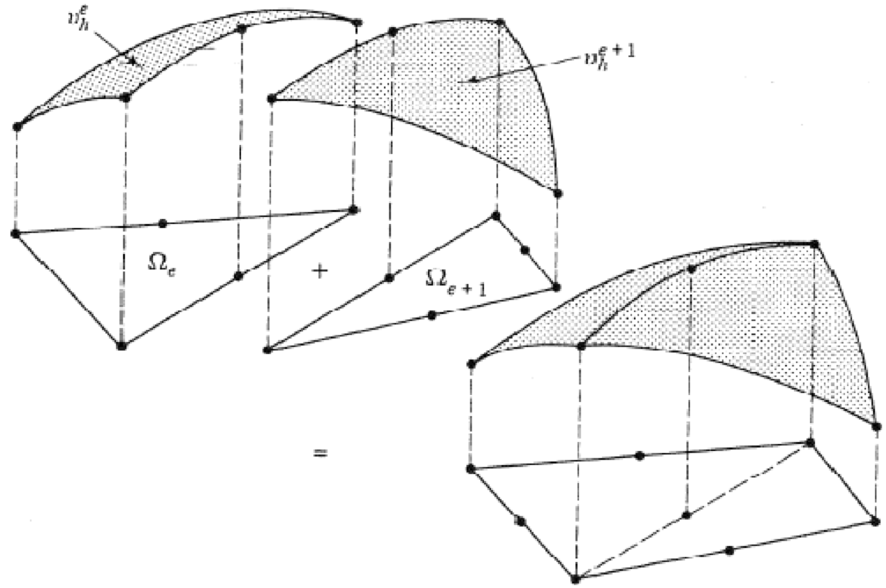
- How about a cubic polynomial with 10 terms? This will require a triangle with 10 nodes.
- The location of the nodes is again determined from the location of entries in the Pascal triangle – one at each vertex, two on each side dividing each side intro 3 equal lengths and one at the centroid.



Triangular elements are continuous over the domain

- These elements produce basis functions that are continuous over the domain, therefore have square integrable first derivatives.

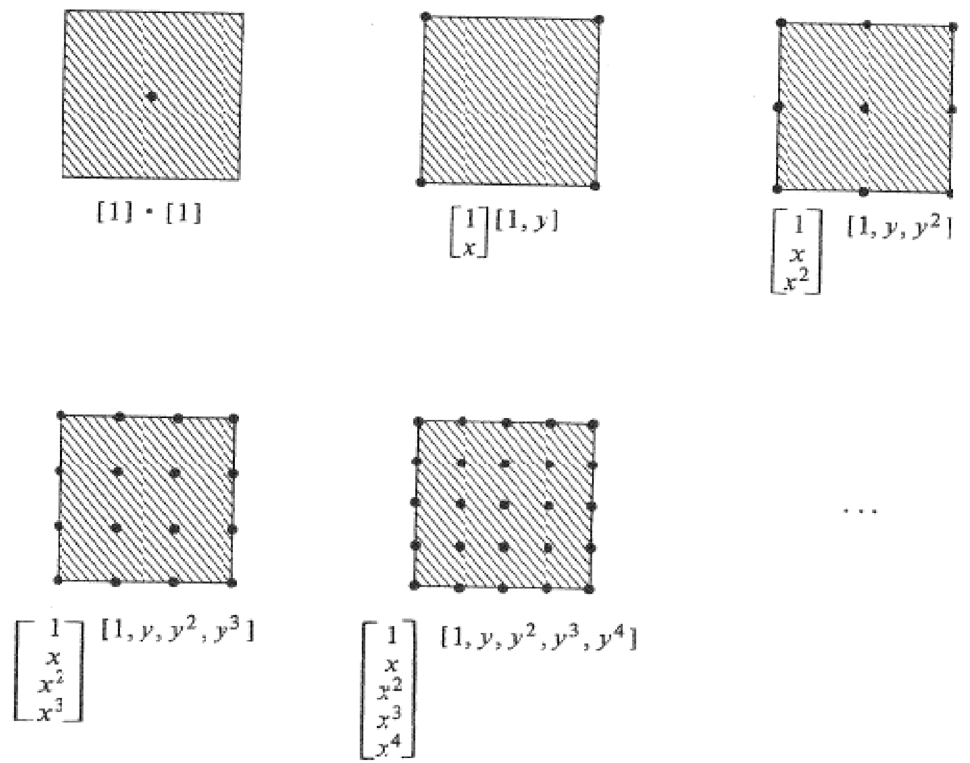
- In the figure you see two neighboring 6-node triangles. The local interpolants v_h^e and v_h^{e+1} are quadratic polynomials that coincide at the 3 nodal points common to both elements.



Rectangular elements

- Various rectangular elements can be produced by tensor products of polynomials in *x* and *y* as shown here.

	1	y	y^2	y^3	y^4	$\cdots$
1	1	y	y^2	y^3	y^4	$\cdots$
x	x	xy	xy^2	xy^3	xy^4	$\cdots$
x^2	x^2	x^2y	x^2y^2	x^2y^3	x^2y^4	$\cdots$
x^3	x^3	x^3y	x^3y^2	x^3y^3	x^3y^4	$\cdots$
x^4	x^4	x^4y	x^4y^2	x^4y^3	x^4y^4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	



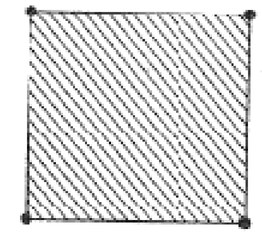
Bilinear polynomials

- The tensor produce of monomials *(1 x)* with the monomials *(1 y)* produces the following matrix:

$$\begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 & y \end{bmatrix} = \begin{bmatrix} 1 & y \\ x & xy \end{bmatrix}$$

- A linear combination of the entries in this matrix produces a local bilinear polynomial approximation.

$$v_h^e(x) = a_1 + a_2 x + a_3 y + a_4 xy$$



- Note that on the site of each of these elements, $v_h^e(x)$ is linear in *x* or *y*.
- These shape functions produce basis functions N_i which are continuous – thus have square integrable 1st derivatives.

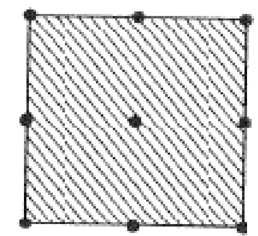
Piece-wise biquadratic basis functions

- The tensor produce of monomials *(1 x x²)* with the monomials *(1 y y²)* produces the following matrix:

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 1 & y & y^2 \end{bmatrix} = \begin{bmatrix} 1 & y & y^2 \\ x & xy & xy^2 \\ x^2 & x^2 y & x^2 y^2 \end{bmatrix}$$

- A linear combination of the entries in this matrix produces a local biquadratic polynomial approximation.

$$v_h^e(x) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^2 y^2$$



$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \quad [1, y, y^2]$$

- Note that on the site of each of these elements, $v_h^e(x)$ is quadratic in *x* or *y*.
- These shape functions produce basis functions N_i which are continuous – thus have square integrable 1st derivatives.

Interpolation error

- We consider interpolation in 2D of a function *g(x,y)* using complete polynomials of order *k*. If the *(k+1)* order derivatives of *g* are bounded in Ω^e, the interpolation error is:

$$\| g - g_h \|_{\infty,\Omega_e} \equiv \max_{(x,y)\in\Omega_e} | g(x,y) - g_h(x,y) | \leq c h_e^{k+1}$$

C is a positive constant and h_e is the `diameter' of Ω^e (largest distance between any 2 points in Ω^e).

- This error estimate holds only if a complete polynomial of order *k* appears in g_h.

Interpolation error

- A similar error estimate can be written for the 1st derivative:

$$\left\| \frac{\partial g}{\partial x} - \frac{\partial g_h}{\partial x} \right\|_{\infty,\Omega_e} \le c_1 h_e^k, \qquad \left\| \frac{\partial g}{\partial y} - \frac{\partial g_h}{\partial y} \right\|_{\infty,\Omega_e} \le c_2 h_e^k,$$

- We define the *H^1*-norm in 2D as follows:

$$\| g \|_1^2 = \int_\Omega \left[g^2 + \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 \right] dxdy$$

- Assuming no discretization error ($\Omega_h = \Omega$) and that *h* is the maximum diameter of all elements in the mesh, we can show

$$\| g - g_h \|_1 \le c_3 h^k, \text{ for } h \text{ sufficiently small}$$

- This estimate is only valid if g_h contains complete polynomials of order *k*.

Interpolation error

$$\| g - g_h \|_1 \leq c_3 h^k, \ for\ h\ sufficiently\ small$$

- Piece-wise linear interpolation on triangles (*k=1*): The error is of order $O(h)$.
- Piece-wise bilinear interpolation: The error is still $O(h)$ even if $v_h^e(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$ contains the quadratic term *xy*. The key idea here is that g_h^e contains complete polynomials of order *k=1* but not *k=2* (the terms x^2 and y^2 are missing)!
- For piece-wise biquadratic polynomial approximation

$$v_h^e(x) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^2 y^2$$

the cubic terms are missing and the error is of the order $O(h^2)$

These approximations have extra terms that allow continuity but do not contribute to the asymptotic rate of convergence of the interpolation error.