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Finite Element Analysis for Mechanical and Aerospace Design

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Function norms

- In preparation for defining error norms in finite element analysis, we next summarize some definitions.
- Consider a function *f(x)* defined in the interval *[a,b]. We define the L_2 norm of this function as follows:*

$$\| f(x) \|_{L_2} = \left(\int_a^b f^2(x)dx \right)^{1/2}$$

Finite element error

- Assume that you know the exact solution in the problem of interest (e.g. our 1D deformation problem). Then the L_2 error in the finite element solution (displacement) is defined as:

$$\| e \|_{L_2} = \| u^h(x) - u^{exact}(x) \|_{L_2} = \left(\int_a^b \left(u^h(x) - u^{exact}(x) \right)^2 dx \right)^{1/2}$$

- Since the FEM solution is defined piecewise over each element, we can write the above error as:

$$\| u^h(x) - u^{exact}(x) \|_{L_2} = \sum_e \left(\int_{x_1^e}^{x_2^e} \left(u^h(x) - u^{exact}(x) \right)^2 dx \right)^{1/2}$$

Finite element error

- The normalized L_2 error in the finite element solution is also used:

$$\frac{\| u^h(x) - u^{exact}(x) \|_{L_2}}{\| u^{exact}(x) \|_{L_2}} = \frac{\left(\int_a^b \left(u^h(x) - u^{exact}(x) \right)^2 dx \right)^{1/2}}{\left(\int_a^b \left(u^{exact}(x) \right)^2 dx \right)^{1/2}}$$

- These definitions are of course not very useful as usually the exact solution is not known!

Energy norm

- In deformation problems, one is often interested in the error in the gradient of the displacement field (e.g. in the strains and stresses).
- A common form to express the error in the gradient is using the energy norm defined:

$$\| e \|_{en} = \| u^h(x) - u^{exact}(x) \|_{en} = \left(\frac{1}{2} \int_a^b E \left(\varepsilon^h(x) - \varepsilon^{exact}(x) \right)^2 dx \right)^{1/2}$$

- Recall that $U = \int_a^b \frac{1}{2} \sigma^h(x) \varepsilon^h(x) dx = \int_a^b \frac{1}{2} E \left(\varepsilon^h(x) \right)^2 dx$ is the strain energy.

Energy norm

- The relative error in the energy is defined similarly as:

$$\bar{e}_{en} = \frac{\| u^h(x) - u^{exact}(x) \|_{en}}{\| u^{exact}(x) \|_{en}} = \frac{\left(\frac{1}{2} \int_a^b E \left(\varepsilon^h(x) - \varepsilon^{exact}(x) \right)^2 dx \right)^{1/2}}{\left(\frac{1}{2} \int_a^b E \left(\varepsilon^{exact}(x) \right)^2 dx \right)^{1/2}}$$

Convergence of the finite element method

- We are interested to know how the error in the FE solution reduces as we refine the FE grid (e.g. if the finite element size is halved).
- If the finite element contains the complete polynomial of order p, then the L_2 error of the displacement varies as follows:

$$\| e \|_{L_2} \leq C h^{p+1}$$

C: constant independent of h but depends on the element polynomial order

p=1: linear elements
p=2: quadratic elements, etc.

Convergence of the finite element method

$$\| e \|_{L_2} \leq C h^{p+1}$$

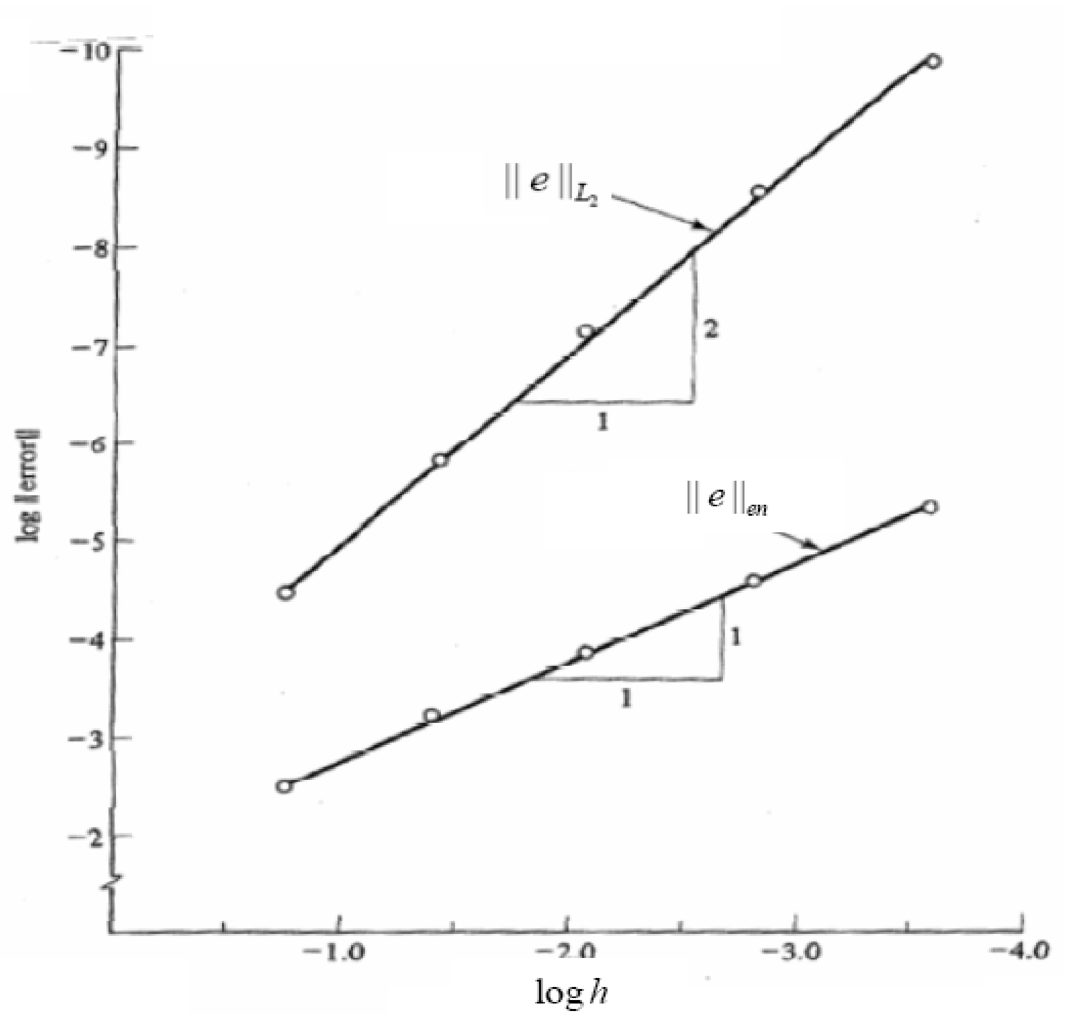
- For linear elements: If you halve the element size, the L_2 error decreases by 1/4
- For quadratic elements: If you halve the element size, the L_2 error decreases by 1/8!
  - The cost of using higher order elements does pay of!

Convergence in the energy norm

$$\| e \|_{en} \leq C h^{p}$$

- For linear elements: If you halve the element size, the *energy* error decreases by 1/2
- For quadratic elements: If you halve the element size, the *energy* error decreases by 1/4!
  - The cost of using higher order elements does pay of!

Log-log plots of the finite element error



Convergence

- The earlier formulas are valid assuming that the actual solution has smooth derivatives up to order $p+1$.
- Assume that the actual solution of our BVP has the property that its derivatives of order λ are square-integrable in Ω but those of order $\lambda+1$ and higher are not, λ being an integer $\lambda > 1$
- As before, we assume that we use shape functions that contain complete polynomials of degree $\leq p$ and a uniform mesh of element of equal size h. Then:

$$\| e \|_{en} \leq C h^{s}$$

$$s = \min(p, \lambda)$$

- Thus the rate of convergence is p if $p<\lambda$ or λ if $\lambda<p$
- Note that for $\lambda<p$, the rate of convergence is independent of p and no improvement is observed by increasing p

Finite element function spaces

- There are an infinite number of functions in *U* and U_0 , i.e. these spaces are of infinite dimension.

- When we represent the weight functions by shape functions, then the space of weight functions U_0^h becomes of finite dimension (equal to the number of nodes excluding those on essential boundary).

- Similarly, the space U^h in which we seek our FE solution becomes finite dimensional.

Finite element function spaces

- While the weak form is exactly equivalent to the strong form for the infinite dimensional spaces U and U_0 , it is only approximately equivalent for the finite dimensional spaces $U^h \subset U$ and $U_0^h \subset U_0$, which are used in the finite element method.

- Therefore, the balance (equilibrium) equation, and the natural boundary conditions that emanate from the weak form are only satisfied approximately.

- In this lecture, we distinguish between the weak forms defined for the exact and finite element solutions.

Finite element error

- Let us consider the weak forms defined for the exact and finite element solutions. For the elasticity problem these equations are given as follows.

$$Find\ u(x) \in U : \int_{\Omega} \frac{dw}{dx} AE \frac{du}{dx} dx = Aw\bar{t}\,|_{\Gamma_t} + \int_{\Omega} bw dx = 0,\ \forall w \in in\, U_0$$

$$Find\ u^h(x) \in U^h : \int_{\Omega} \frac{dw^h}{dx} AE \frac{du^h}{dx} dx = Aw^h\bar{t}\,|_{\Gamma_t} + \int_{\Omega} bw^h dx = 0,\ \forall w^h \in in\, U_0^h$$

- Let $e = u - u^h$ (finite element error). Subtracting the two weak forms gives (note the 1st eq. is also valid for $\forall w^h \in in\, U_0^h$, since $U_0^h \subset U_0$):

$$\int_{\Omega} \frac{dw^h}{dx} AE \frac{de}{dx} dx = 0,\ \forall w^h \in in\, U_0^h$$

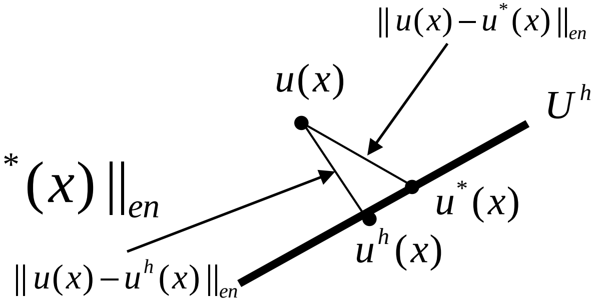
Finite element error estimation

- How do we estimate the error when the exact solution is not known?
- Let us consider the finite element spaces

$$U^h \subset U, U_0^h \subset U_0.$$

- We will prove that $u^h(x)$ minimizes the energy norm of $\| u(x) - u^*(x) \|_{en}$, *for any* $u^*(x) \in U^h$. Here, $u(x)$ is the exact solution.

$$\| u(x) - u^h(x) \|_{en} = \min_{u^*(x) \in U^h} \| u(x) - u^*(x) \|_{en}$$



Finite element error estimation

$$\| u(x) - u^*(x) \|_{en}^2 = \| \underbrace{\left(u(x) - u^h(x) \right)}_{e} + \underbrace{(u^h(x) - u^*(x))}_{w^h \in U_0^h} \|_{en}^2$$

- Note that since $u^h(x)$ and $u^*(x)$ satisfy essential boundary conditions, it follows that $u^*(x) - u^h(x) \equiv w^h \in U_0^h$ and thus:

$$\| u(x) - u^*(x) \|_{en}^2 = \| e + w^h \|_{en}^2 = \| e \|_{en}^2 + \| w^h \|_{en}^2 + 2\int_{\Omega} \frac{dw^h}{dx} AE \frac{de}{dx} dx$$

- However, we already have shown that the last integral vanishes:

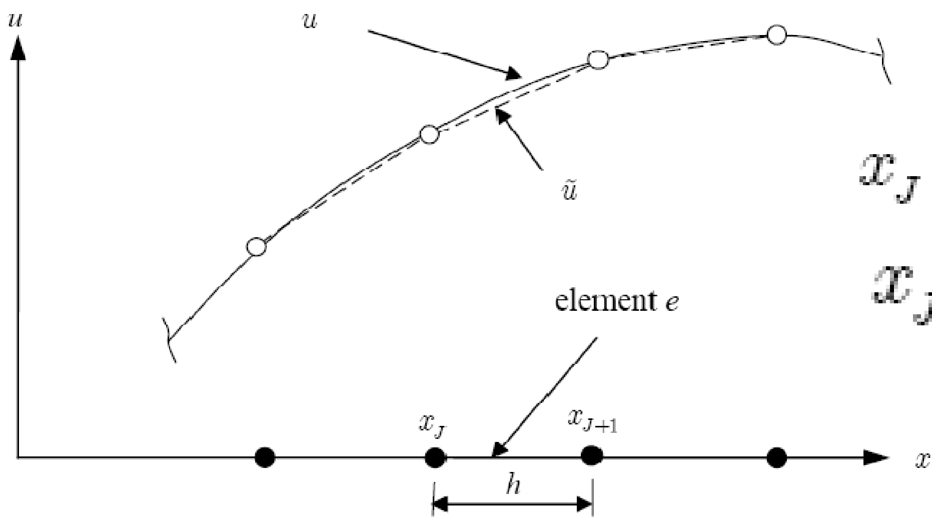
$$\| u(x) - u^*(x) \|_{en}^2 = \| e \|_{en}^2 + \| w^h \|_{en}^2 \geq \| e \|_{en}^2$$

Finite element error estimation

$$\| u(x) - u^h(x) \|_{en} \le \| u(x) - u^*(x) \|_{en}$$

- Thus to compute the finite element error, we need to find some function $u^*(x) \in U$ for which the error $\| u(x) - u^*(x) \|_{en}$ is easier to compute.
- We will use as $u^*(x) \in U$ the finite element interpolant $\tilde{u}(x) \in U^h$ of *u(x),* i.e. a function in U^h that interpolates *u(x)* between nodes (i.e. it is equal to the exact solution at the finite element nodes, $\tilde{u}(x_j) = u(x_j)$).

Finite element interpolant



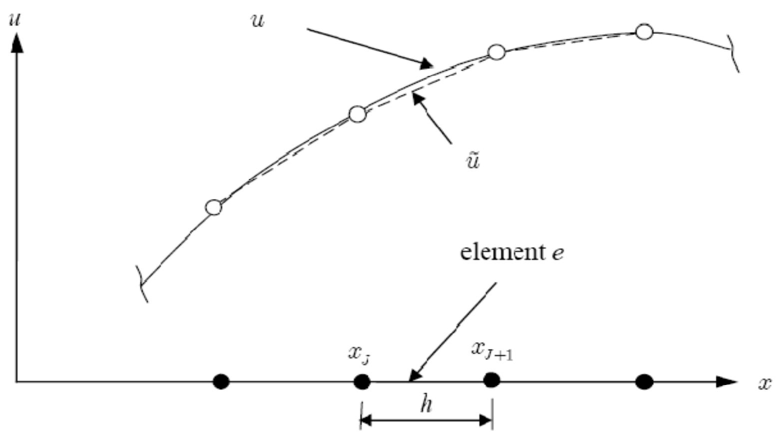
$$x_J = (e-1)h$$

$$x_{J+1} = eh$$

- The derivative of the interpolation function $\tilde{u}_{,x}$ in element e is given by: $$\tilde{u}_{,x} = \frac{\tilde{u}(x_{j+1}) - \tilde{u}(x_j)}{x_{j+1} - x_j}$$
- By the mean value theorem there is a c, $x_j \le c \le x_{j+1}$:

$$\tilde{u}_{,x} = u_{,x}(c)$$

Interpolation error



$$x_J = (e-1)h$$

$$x_{J+1} = eh$$

$$\tilde{u}_{,x} = u_{,x}(c)$$

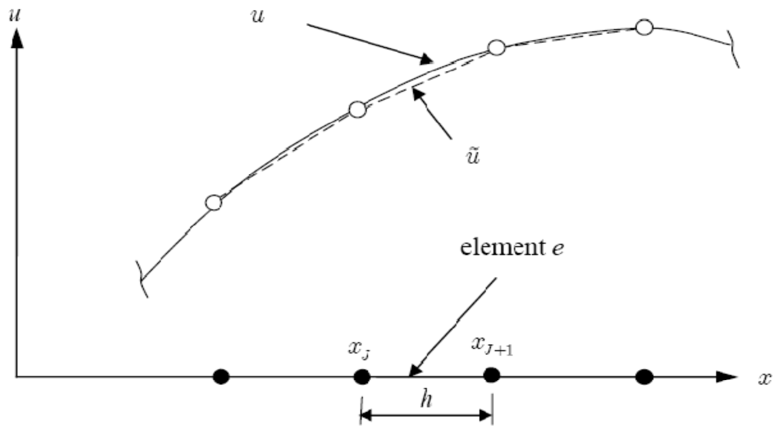
- Expand the derivative of the exact solution using a Taylor's expansion:

$$u_{,x}(x) = u_{,x}(c) + (x-c)u_{,xx}(\xi),\ c \le \xi \le x.$$

- Subtracting the above two Eqs and assuming $|u_{,xx}(\xi)| \le a$:

$$u_{,x}(x) - \tilde{u}_{,x}(x) = (x-c)u_{,xx}(\xi) \Rightarrow |\tilde{e}^{e}_{,x}| = \underbrace{|u_{,x}(x) - \tilde{u}_{,x}(x)|}_{\text{Interpolation error}} = |(x-c)u_{,xx}(\xi)| \le a\,|x - x_j| \le ah$$

Finite element error for linear elements



$$x_J = (e-1)h$$

$$x_{J+1} = eh$$

$$|\tilde{e}^e_{,x}| \le ah$$

- The interpolation error in the energy norm is then:

$$|\tilde{e}|^e_{en} = \frac{1}{2}\int_\Omega AE(\tilde{e}^e_{,x})^2\,dx = \frac{1}{2}\sum_{e=1}^{n}\int_{(e-1)h}^{eh} A^e E^e (\tilde{e}^e_{,x})^2\,dx \le \frac{1}{2} nhK(ha)^2$$

where we assumed $A^e E^e \le K$

- Denoting *nh=L*, and recalling that the energy norm of the finite element solution error is less than or equal to the energy norm of the interpolation error: $\|e\|_{en} \le \sqrt{\frac{1}{2}KLa^2h^2} = Ch$