MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

Cornell University, Fall 2009

Nicholas Zabaras
Materials Process Design and Control Laboratory
Sibley School of Mechanical and Aerospace Engineering
101 Rhodes Hall
Cornell University
Ithaca, NY 14853-3801

Development of discrete equations

- Consider the elastic bar problem with arbitrary BCs.

$$\int_{\Omega} \underbrace{\left(\frac{dw}{dx}\right)^T}_{This\ is\ a\ scalar} EA\frac{du}{dx}dx - \underbrace{w^T}_{This\ is\ a\ scalar} A\bar{t}\,|_{\Gamma_t} - \int_{\Omega} \underbrace{w^T}_{This\ is\ a\ scalar} bdx = 0, \forall w(x)\ in\ 0 < x < L,\ with\ w \in U_0$$

- Note that we added the transpost *()^T* on *w* terms for allowing us to do easy matrix operations at a later time.

  At this time, this has no effects as *w* is a scalar!

- Having selected the FE mesh and introducing smooth approximation functions over each element, we can write:

$$\sum_e \int_{\Omega^e} \left(\frac{dw^e}{dx}\right)^T E^e A^e \frac{du^e}{dx}dx - w^{e^T} A^e \bar{t}\,|_{\Gamma_t} - \sum_e \int_{\Omega^e} w^{e^T} b^e dx = 0, \forall\ w \in U_0$$

Development of discrete equations

$$\sum_e \int_{\Omega^e} \underbrace{\left(\frac{dw^e}{dx}\right)^T}_{scalar} E^e A^e \underbrace{\frac{du^e}{dx}}_{scalar} dx - \underbrace{w^{e^T}}_{scalar} A^e \bar{t}\,|_{\Gamma_t} - \sum_e \int_{\Omega^e} \underbrace{w^{e^T}}_{scalar} b^e dx = 0, \ \forall\ w_F \in U_0$$

- We use the same approximations for w^e and u^e :

$$\underbrace{u^e}_{scalar} = \underbrace{\left[N^e\right]}_{row\ vector} \underbrace{\left\{u^e\right\}}_{\substack{column\\ vector}}, \quad \underbrace{\frac{du^e}{dx}}_{scalar} = \underbrace{\left[B^e\right]}_{row\ vector} \left\{u^e\right\} = \underbrace{\left[B^e\right]}_{row\ vector} \underbrace{\left\{d^e\right\}}_{\substack{Nodal\\ DOF}}$$

$$\underbrace{w^e = w^{eT}}_{Scalars} = \left\{w^e\right\}^T \left[N^e\right]^T, \quad \left(\frac{dw^e}{dx}\right)^T = \left\{w^e\right\}^T \left[B^e\right]^T$$

- Substitution into the weak form gives:

$$\sum_e \left\{w^e\right\}^T \left\{ \underbrace{\int_{\Omega^e} \left[B^e\right]^T A^e E^e \left[B^e\right] dx}_{Element\ stiffness} \left\{d^e\right\} - \sum_e \left(\underbrace{\int_{\Omega^e} \left[N^e\right]^T b^e dx}_{\substack{Element\ Load\\ Distributed}} + \underbrace{\left[N^e\right]^T A^e \bar{t}\,|_{\Gamma_t}}_{\substack{Element\ load\\ Concentrated}} \right) \right\} = 0, \ \forall\ \left\{w_F\right\}$$

Development of discrete equations

$$\sum_e \left\{w^e\right\}^T \left\{ \int_{\Omega^e} \left[B^e\right]^T A^e E^e \left[B\right]^e dx \left\{d^e\right\} - \sum_e \int_{\Omega^e} \left[N^e\right]^T b^e dx - \left[N^e\right]^T A^e \bar{t} \,|_{\Gamma_t} \right\} = 0, \forall \left\{w_F\right\}$$

$$\left[K^e\right] = \int_{\Omega^e} \left[B^e\right]^T A^e E^e \left[B^e\right] dx$$

$$\underbrace{\left\{f^e\right\}}_{\text{Column vector}} = \underbrace{\int_{\Omega^e} \left[N^e\right]^T b^e dx}_{\left\{f^e_\Omega\right\}} + \underbrace{\left(\left[N^e\right]^T A^e \bar{t}\right)|_{\Gamma_t}}_{\left\{f^e_\Gamma\right\}}$$

- The element and global matrices are related:

$$\left\{w^e\right\} = \left[L^e\right]\left\{w\right\}$$
$$\left\{d^e\right\} = \left[L^e\right]\left\{d\right\}$$

- The global stiffness and load now take the form:

$$\left\{w\right\}^T \left(\underbrace{\sum_e \left[L^e\right]^T \left[K^e\right]\left[L^e\right]}_{[K]} \left\{d\right\} - \underbrace{\sum_e \left[L^e\right]^T \left\{f^e\right\}}_{\{F\}} \right) = 0, \forall \left\{w_F\right\}$$

Partitioning of the global solution

- As we did in earlier lectures, we partition the solution and weight function to account for essential boundary conditions:

$$d = \begin{Bmatrix} \overline{d}_E \\ d_F \end{Bmatrix}, \; w = \begin{Bmatrix} w_E \\ w_F \end{Bmatrix} = \begin{Bmatrix} 0 \\ w_F \end{Bmatrix}$$

- So in our discretized weak form the statement for every $w \in U_0$ needs to be translated to for every $\{w_F\}$.

Development of discrete equations

$$\{w\}^T \underbrace{\left(\underbrace{\sum_e [L^e]^T [K^e][L^e]}_{[K]} \{d\} - \underbrace{\sum_e [L^e]^T \{f^e\}}_{\{F\}} \right)}_{r} = 0, \forall \{w_F\}$$

- Using the partition *w* and similar partition for the residual *r*, the above equation leads to:

$$[w_E \quad w_F]^T \begin{Bmatrix} r_E \\ r_F \end{Bmatrix} = \{w_E\}^T \{r_E\} + \{w_F\}^T \{r_F\} = 0, \forall \{w_F\}$$

- Since $\{w_E\} = 0$, this leads to: $\{r_F\} = 0,$ and we can summarize:

$$\{r\} = \begin{Bmatrix} r_E \\ 0 \end{Bmatrix} = \begin{bmatrix} K_E & K_{EF} \\ K_{EF}^T & K_F \end{bmatrix} \begin{Bmatrix} \overline{d}_E \\ d_F \end{Bmatrix} - \begin{Bmatrix} f_E \\ f_F \end{Bmatrix} \Rightarrow \begin{bmatrix} K_E & K_{EF} \\ K_{EF}^T & K_F \end{bmatrix} \begin{Bmatrix} \overline{d}_E \\ d_F \end{Bmatrix} = \begin{Bmatrix} f_E + r_E \\ f_F \end{Bmatrix}$$

Discrete finite element equations

$$\begin{bmatrix} K_E & K_{EF} \\ K_{EF}^T & K_F \end{bmatrix} \begin{Bmatrix} \bar{d}_E \\ d_F \end{Bmatrix} = \begin{Bmatrix} f_E + r_E \\ f_F \end{Bmatrix}$$

- The unknown displacements can be computed from:

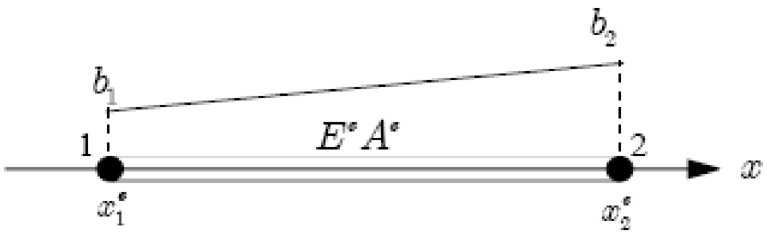
$$\left[K_F\right]\left\{d_F\right\} = \left\{f_F\right\} - \left[K_{EF}\right]^T \left\{\bar{d}_E\right\}$$

- And the reaction forces from:

$$\left[r_E\right] = \left[K_E\right]\left\{\bar{d}_E\right\} + \left[K_{EF}\right]\left\{d_F\right\} - \left\{f_E\right\}$$

- The displacement field in each element is: $u^e(x) = \left[N^e(x)\right]\left\{d^e\right\}$ and the stresses $\sigma^e(x) = E^e(x)\left[B^e(x)\right]\left\{d^e\right\},$ where: $\left\{d^e\right\} = \left[L^e\right]\left\{d\right\}$

Element stiffness matrix



$$\left[N^e\right] = \left[\frac{x_2^e - x}{x_2^e - x_1^e} \quad \frac{x - x_1^e}{x_2^e - x_1^e}\right] = \frac{1}{L^e}\left[x_2^e - x \quad x - x_1^e\right]$$

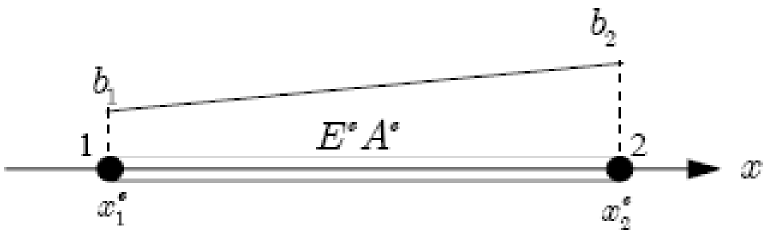
$$\left[B^e\right] = \left[\frac{dN^e}{dx}\right] = \left[-\frac{1}{x_2^e - x_1^e} \quad \frac{1}{x_2^e - x_1^e}\right] = \frac{1}{L^e}\left[-1 \quad 1\right]$$

- The element stiffness is computed as:

$$\left[K^e\right] = \int_{\Omega^e} \left[B^e\right]^T A^e E^e \left[B^e\right] dx = \int_{x_1^e}^{x_2^e} \frac{1}{L^e} \begin{bmatrix} -1 \\ 1 \end{bmatrix} A^e E^e \frac{1}{L^e}\left[-1 \quad 1\right] dx = \frac{A^e E^e}{L^{e2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \left[-1 \quad 1\right] \int_{x_1^e}^{x_2^e} dx \Rightarrow$$

$$\left[K^e\right] = \frac{A^e E^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element load matrix



$$\left[N^e\right] = \left[\frac{x_2^e - x}{x_2^e - x_1^e} \quad \frac{x - x_1^e}{x_2^e - x_1^e}\right] = \frac{1}{L^e}\left[x_2^e - x \quad x - x_1^e\right]$$

$$\left[B^e\right] = \left[\frac{dN^e}{dx}\right] = \left[-\frac{1}{x_2^e - x_1^e} \quad \frac{1}{x_2^e - x_1^e}\right] = \frac{1}{L^e}\left[-1 \quad 1\right]$$

- The force term f_Ω^e is:

Note we use the FE interpolant of the given force!

$$\left\{f_\Omega^e\right\} = \int_{x_1^e}^{x_2^e} \left[N^e\right]^T \underbrace{b^e}_{Scalar} dx = \int_{x_1^e}^{x_2^e} \left[N^e\right]^T \left[N^e\right] \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} dx$$

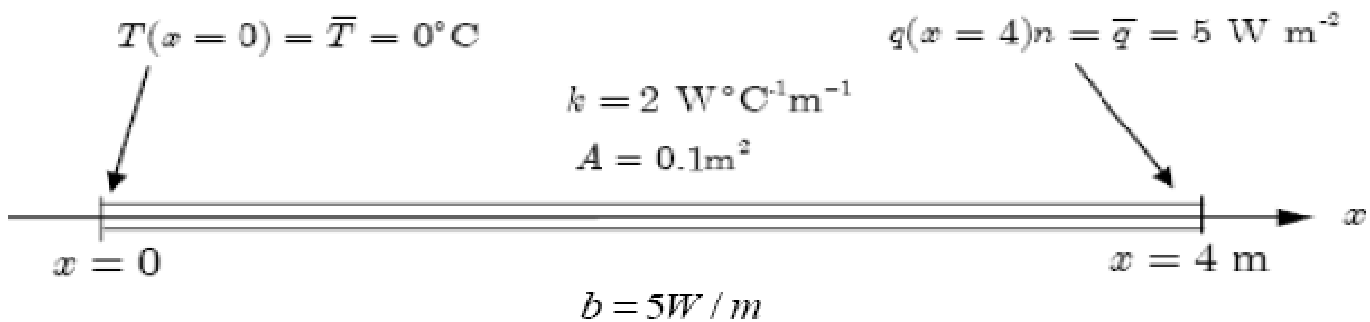
$$\left\{f_\Omega^e\right\} = \frac{1}{L^{e2}} \int_{x_1^e}^{x_2^e} \begin{bmatrix} (x_2^e - x)^2 & (x_2^e - x)(x - x_1^e) \\ (x_2^e - x)(x - x_1^e) & (x - x_1^e)^2 \end{bmatrix} dx \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

$$\left\{f_\Omega^e\right\} = \frac{L^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

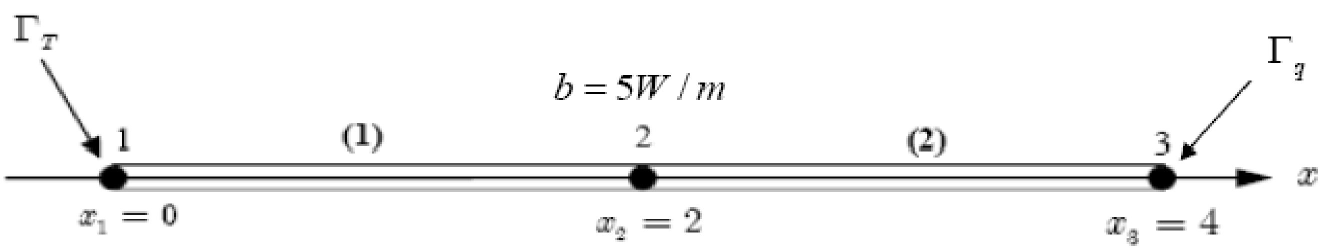
For equally distributed load $b_1 = b_2$, $\left\{f_\Omega^e\right\} = \frac{L^e b}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

The total load ($L^e b$) is equally distributed at the 2 nodes (as expected!)

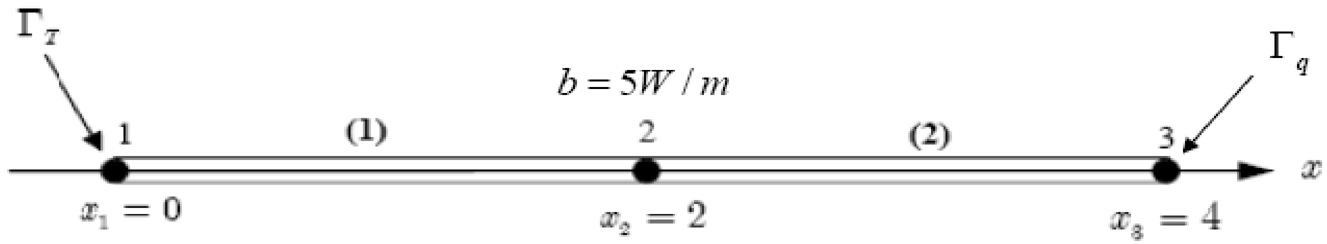
Example problem: Heat conduction



- For the heat conduction problem shown, we will compute the temperature and heat flux distribution using 2 linear finite elements.



Example problem: Heat conduction

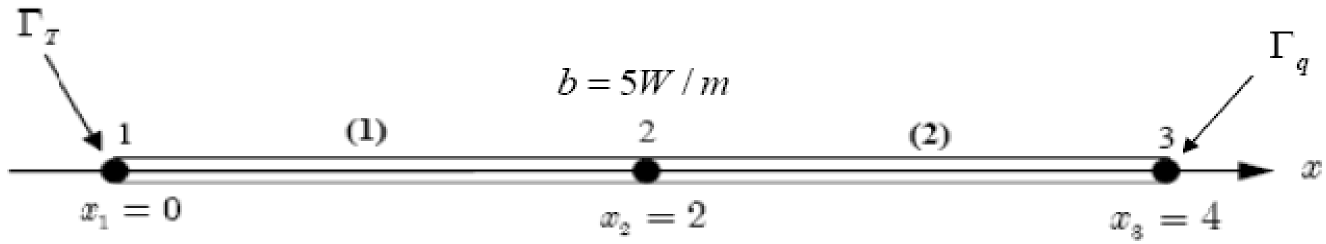


- The element stiffness matrix and load vectors for heat conduction are given as:

$$\left[K^e\right] = \int_{\Omega^e} \left[B^e\right]^T A^e k^e \left[B^e\right] dx$$

$$\left\{f^e\right\} = \int_{\Omega^e} \left[N^e\right]^T b\,dx - \left(\left[N^e\right]^T A^e \bar{q}\right)\big|_{\Gamma_q}$$

Element 1

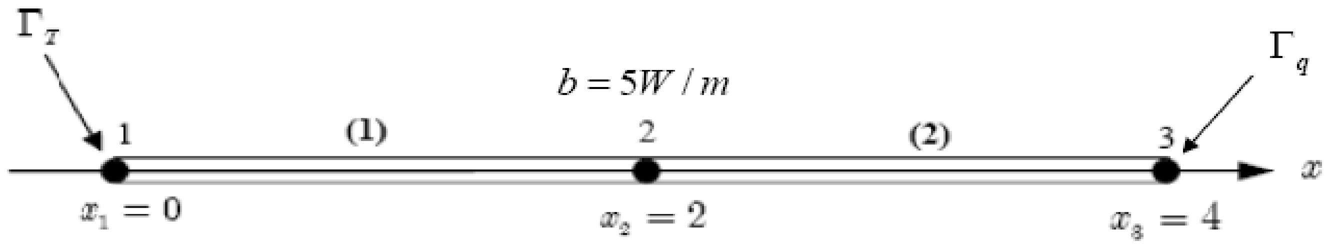


$$\left[N^1\right] = \left[\frac{x_2^1 - x}{x_2^1 - x_1^1} \quad \frac{x - x_1^1}{x_2^1 - x_1^1}\right] = \left[\frac{2-x}{2} \quad \frac{x}{2}\right] \qquad \left[B^1\right] = \left[\frac{dN^1}{dx}\right] = \left[-\frac{1}{x_2^1 - x_1^1} \quad \frac{1}{x_2^1 - x_1^1}\right] = \frac{1}{2}\left[-1 \quad 1\right]$$

$$\left[K^1\right] = \int_{\Omega^2} \left[B^1\right]^T A^1 k^1 \left[B^1\right] dx = \int_0^2 \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} 0.2 \frac{1}{2}\left[-1 \quad 1\right] dx = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$\left\{f^1\right\} = \int_{\Omega^e} \left[N^1\right]^T b\,dx - \cancel{\left(\left[N^1\right]^T A^1 \bar{q}\right)\Big|_{\Gamma_q}} = \frac{L^1 q}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

Element 2

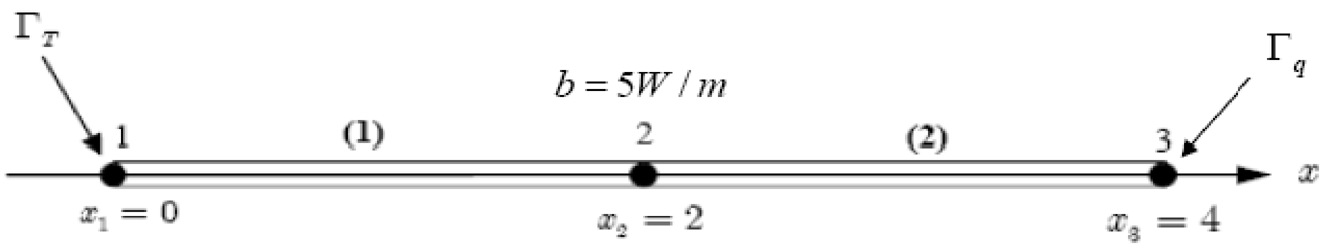


$$\left[N^2\right] = \left[\frac{x_2^1 - x}{x_2^1 - x_1^1} \quad \frac{x - x_1^1}{x_2^1 - x_1^1}\right] = \left[\frac{4-x}{2} \quad \frac{x-2}{2}\right] \qquad \left[B^2\right] = \left[\frac{dN^2}{dx}\right] = \left[-\frac{1}{x_2^2 - x_1^2} \quad \frac{1}{x_2^2 - x_1^2}\right] = \frac{1}{2}\left[-1 \quad 1\right]$$

$$\left[K^2\right] = \int_{\Omega^2} \left[B^2\right]^T A^2 k^2 \left[B^2\right] dx = \int_0^2 \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} 0.2 \frac{1}{2}\left[-1 \quad 1\right] dx = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$$\left\{f^2\right\} = \int_{\Omega^2} \left[N^2\right]^T b dx - (\left[N^2\right]^T A^2 \bar{q})|_{\Gamma_q} = \frac{L^2 q}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \dfrac{4-x}{2} \\ \dfrac{x-2}{2} \end{bmatrix}_{x=4} 0.1 \bullet 5 = \begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4.5 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array}$$

Assembly

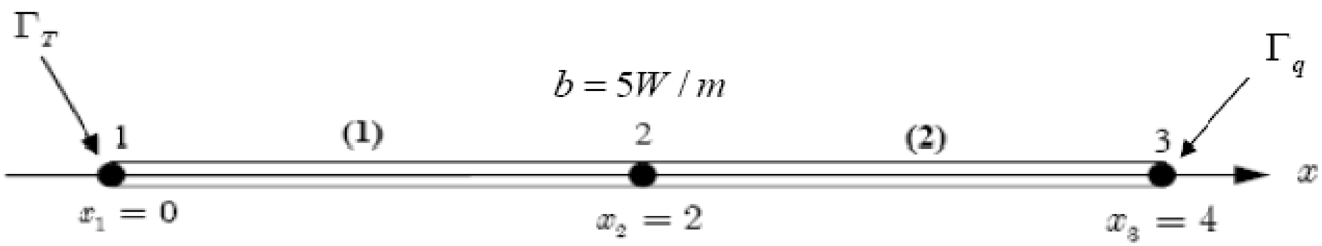


$$[K] = \sum_e \left[L^e\right]^T \left[K^e\right]\left[L^e\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$[K] = \begin{bmatrix} 0.1 & -0.1 & 0.0 \\ -0.1 & 0.1+0.1 & -0.1 \\ 0.0 & -0.1 & 0.1 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.1 & -0.1 & 0.0 \\ -0.1 & 0.2 & -0.1 \\ 0.0 & -0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

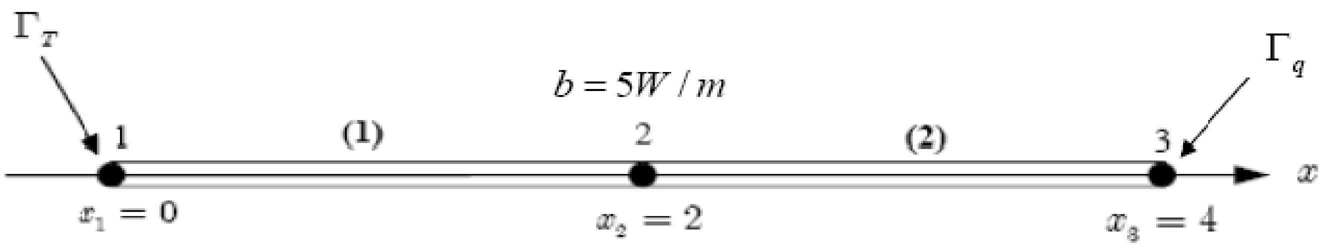
Assembly



$$\{F\} = \sum_e \left[L^e\right]^T \left\{f^e\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4.5 \end{bmatrix} \Rightarrow$$

$$\{F\} == \begin{array}{c} \begin{bmatrix} 5 \\ 10 \\ 4.5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Assembly

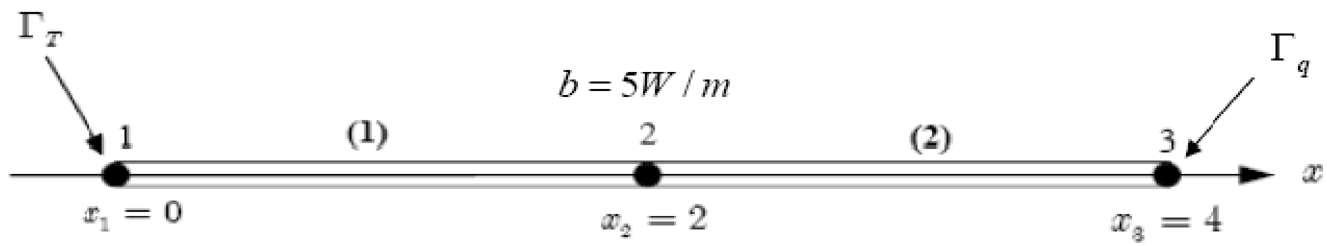


$$\begin{bmatrix} 0.1 & -0.1 & 0.0 \\ -0.1 & 0.2 & -0.1 \\ 0.0 & -0.1 & 0.1 \end{bmatrix} \begin{Bmatrix} T_1 = 0 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 5 + r_1 \\ 10 \\ 4.5 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 10 \\ 4.5 \end{bmatrix} \Rightarrow \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 145 \\ 190 \end{Bmatrix}$$

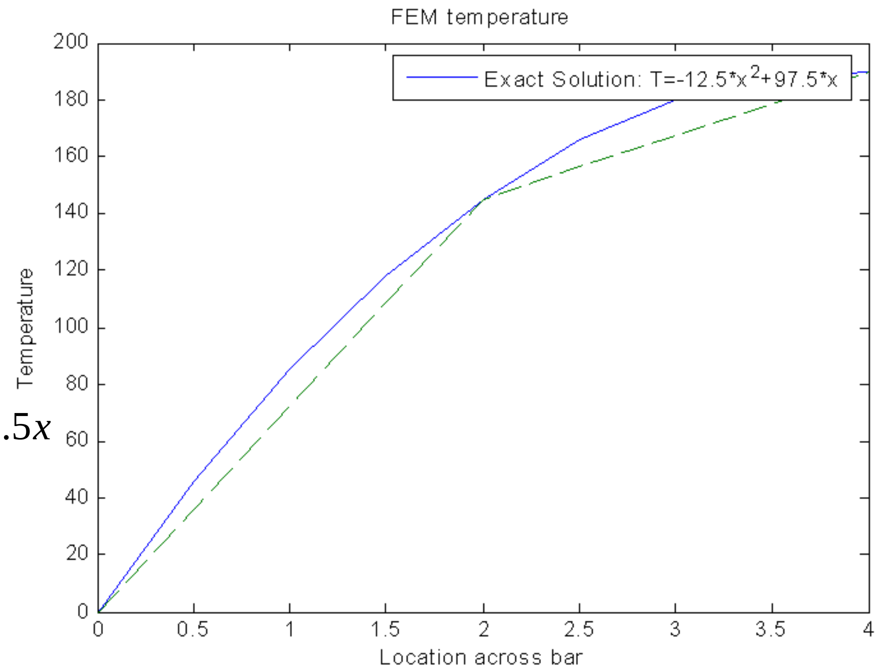
$$\begin{bmatrix} -0.1 & 0.0 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 5 + r_1 \end{bmatrix} \Rightarrow r_1 = -19.5$$

Postprocessing: Temperature

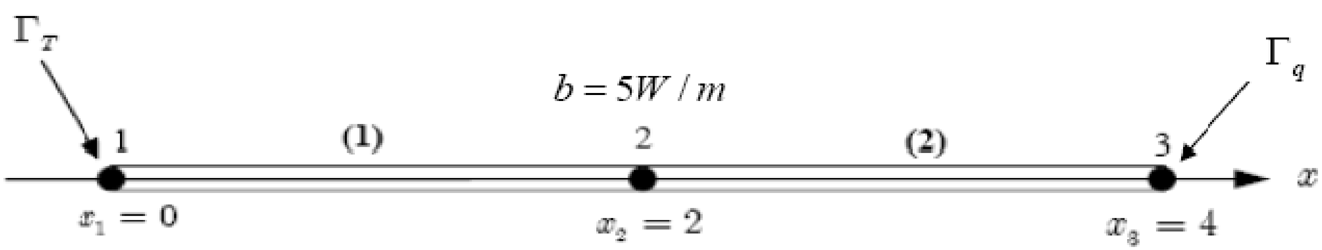


$$T^1 = \left[N^1\right]\left[L^1\right]\left\{d^1\right\} = \begin{bmatrix} \frac{2-x}{2} & \frac{x}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 145 \\ 190 \end{bmatrix} = 72.5x$$

$$T^2 = \left[N^2\right]\left[L^2\right]\left\{d\right\} = \begin{bmatrix} \frac{4-x}{2} & \frac{x-2}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 145 \\ 190 \end{bmatrix} = 100 + 22.5x$$

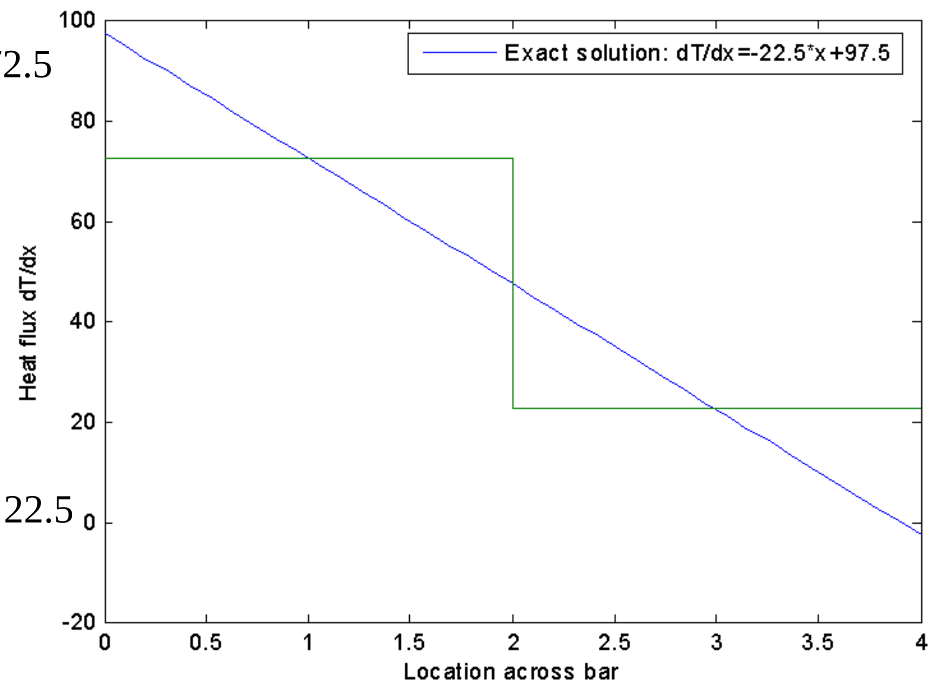


Postprocessing

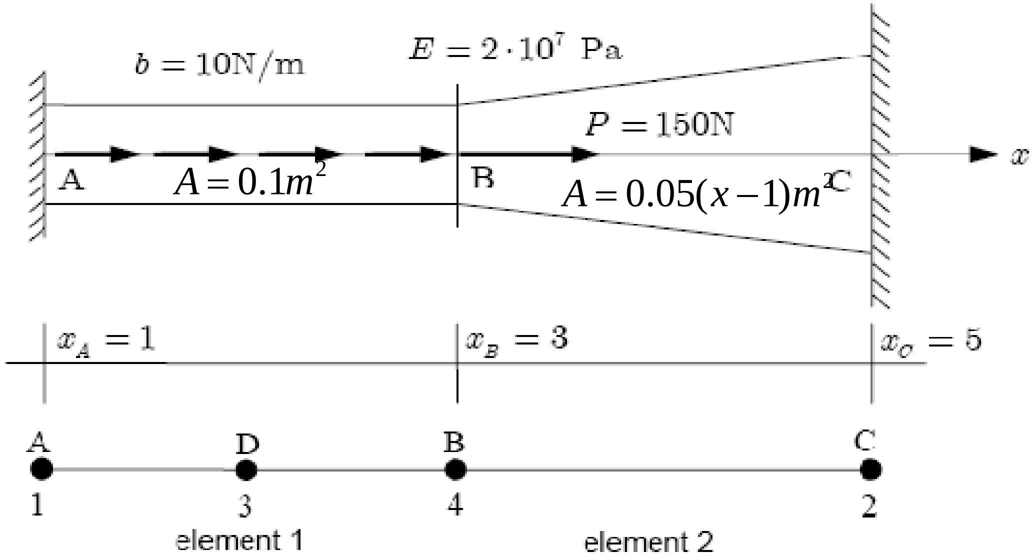


$$\frac{dT^1}{dx} = \left[B^1\right]\left[L^1\right]\left[d^1\right] = \frac{1}{2}\begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\145\\190\end{bmatrix} = 72.5$$

$$\frac{dT^2}{dx} = \left[B^2\right]\left[L^2\right]\left\{d^2\right\} = \frac{1}{2}\begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\145\\190\end{bmatrix} = 22.5$$

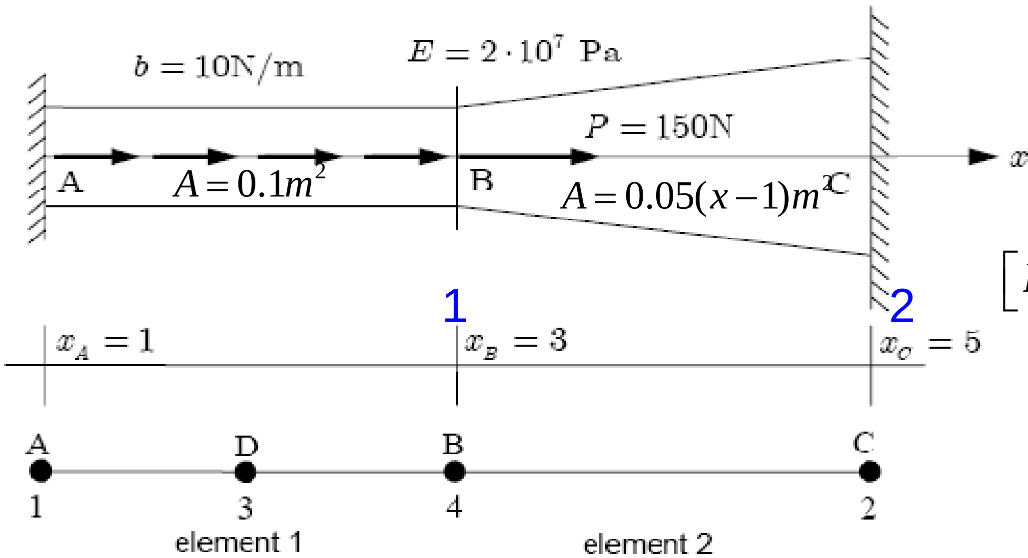


Example problem



- We need to compute the global stiffness matrix and load vector and solve for the displacements and stresses. We consider a 3-node element from A to B and one 2-node element from B to C.

Element 2



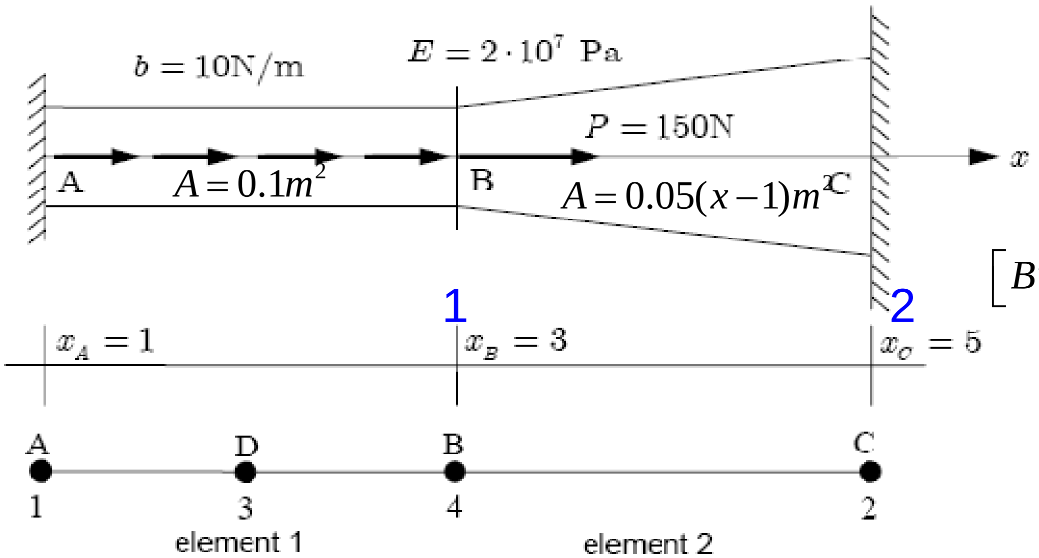
$$\left[N^2\right]=\left[\frac{x_2^2-x}{x_2^2-x_1^2} \quad \frac{x-x_1^2}{x_2^2-x_1^2}\right]=\frac{1}{2}\left[5-x \quad x-3\right]$$

$$\left[B^2\right]=\left[\frac{dN^2}{dx}\right]=\left[-\frac{1}{x_2^2-x_1^2} \quad \frac{1}{x_2^2-x_1^2}\right]=\frac{1}{2}\left[-1 \quad 1\right]$$

$$\left[K^2\right]=\int_{x_B}^{X_D}\left[B^2\right]^T A^2E^2\left[B^2\right]dx=\int_3^3\frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}0.05(x-1)(2e^7)\frac{1}{2}\left[-1 \quad 1\right]dx$$

$$\left[K^2\right]=\int_{x_B}^{X_D}\left[B^2\right]^T A^2E^2\left[B^2\right]dx=10^6\begin{array}{cc} 4 & 2 \\ \left[\begin{matrix}1.5 & -1.5\\-1.5 & 1.5\end{matrix}\right] & \begin{matrix}4\\2\end{matrix}\end{array}$$

Element 2

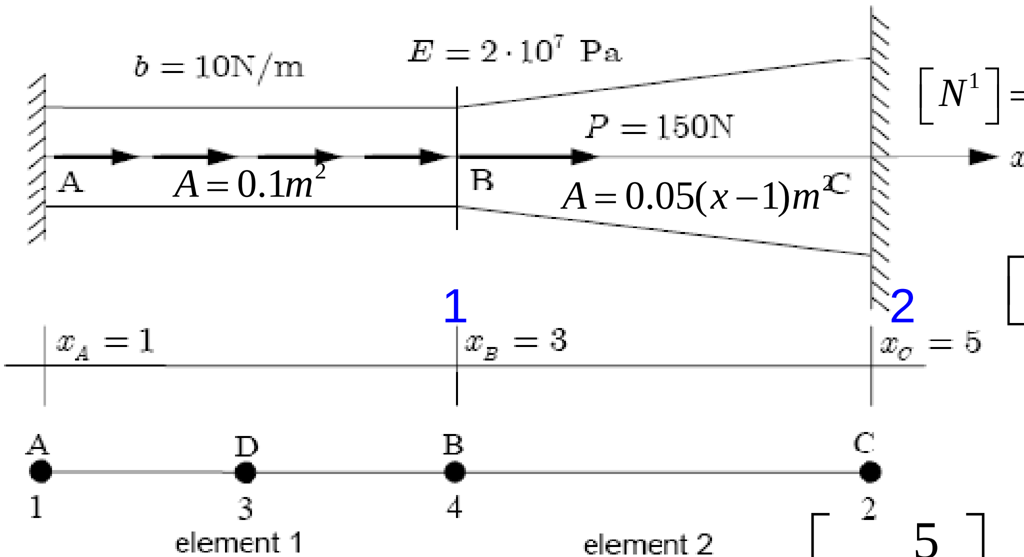


$$\left[N^2\right]=\left[\frac{x_2^2-x}{x_2^2-x_1^2}\quad \frac{x-x_1^2}{x_2^2-x_1^2}\right]=\frac{1}{2}\left[5-x \quad x-3\right]$$

$$\left[B^2\right]=\left[\frac{dN^2}{dx}\right]=\left[-\frac{1}{x_2^2-x_1^2}\quad \frac{1}{x_2^2-x_1^2}\right]=\frac{1}{2}\left[-1 \quad 1\right]$$

$$\left\{f^2\right\}=\cancel{\int_{\Omega^e}\left[N^2\right]^T b dx}+(N^{2^T}P)\,|_{\Gamma_t}=\frac{1}{2}\begin{bmatrix}5-x\\ x-3\end{bmatrix}_{x=3}150=\begin{array}{c c}\begin{bmatrix}150\\ 0\end{bmatrix} & \begin{array}{c}4\\ 2\end{array}\end{array}$$

Element 1



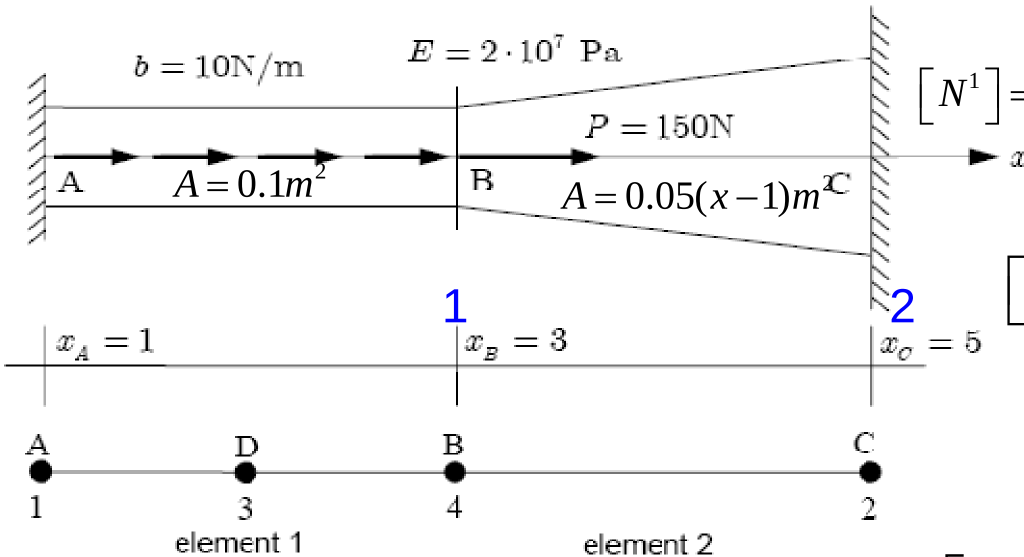
$$\left[N^1\right] = \left[\frac{1}{2}(x-2)(x-3) \quad -(x-1)(x-3) \quad \frac{1}{2}(x-1)(x-2)\right]$$

$$\left[B^1\right] = \left[\frac{dN^1}{dx}\right] = \left[x - \frac{5}{2} \quad 4-2x \quad x-\frac{3}{2}\right]$$

$$\left[K^1\right] = \int_{x_A}^{X_B} \left[B^1\right]^T A^1 E^1 \left[B^1\right] dx = \int_1^3 \begin{bmatrix} x-\frac{5}{2} \\ 4-2x \\ x-\frac{3}{2} \end{bmatrix} 0.1(2e^7)\frac{1}{2}\left[x-\frac{5}{2} \quad 4-2x \quad x-\frac{3}{2}\right] dx$$

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \left[K^1\right] = 10^6 & \begin{bmatrix} 2.333 & -2.6667 & 0.333 \\ -2.6667 & 5.333 & -2.6667 \\ 0.333 & -2.6667 & 2.333 \end{bmatrix} \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \end{array}$$

Element 1

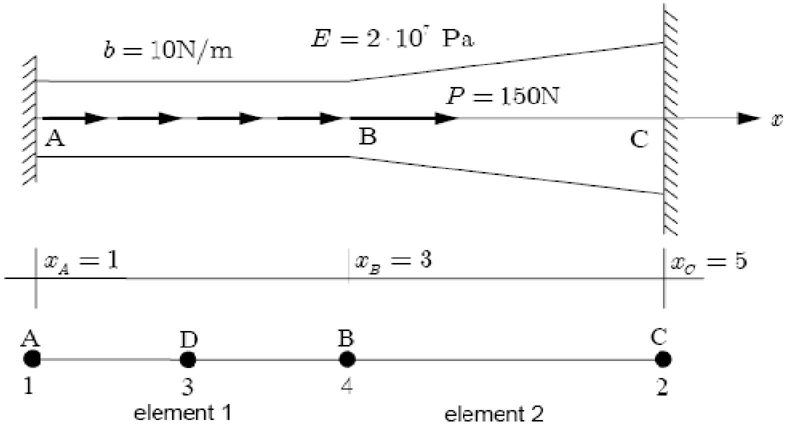


$$\left[N^1\right] = \left[\frac{1}{2}(x-2)(x-3) \quad -(x-1)(x-3) \quad \frac{1}{2}(x-1)(x-2)\right]$$

$$\left[B^1\right] = \left[\frac{dN^1}{dx}\right] = \left[x - \frac{5}{2} \quad 4 - 2x \quad x - \frac{3}{2}\right]$$

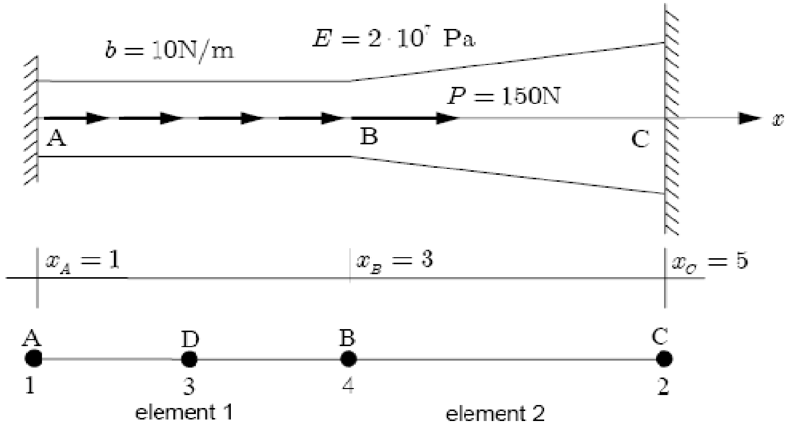
$$\left\{f^1\right\} = \int_{\Omega^1} \left[N^1\right]^T b\,dx + \cancel{\left(\left[N^1\right]^T A^1 P\right)\Big|_{\Gamma_t}} = \int_1^3 \begin{bmatrix} \frac{1}{2}(x-2)(x-3) \\ -(x-1)(x-3) \\ \frac{1}{2}(x-1)(x-2) \end{bmatrix} 10dx = \begin{array}{cc} \begin{bmatrix} 3.333 \\ 13.333 \\ 3.333 \end{bmatrix} & \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \end{array}$$

Assembly



$$[K]=10^6\left[\begin{array}{cc|cc}2.333 & 0 & -2.6667 & 0.333\\ 0 & 1.5 & 0 & -1.5\\ \hline -2.6667 & 0 & 5.333 & -2.6667\\ 0.333 & -1.5 & -2.6667 & 3.8333\end{array}\right]\left\{\begin{array}{c}u_1=0\\ u_2=0\\ \hline u_3\\ u_4\end{array}\right\}=\left\{\begin{array}{c}r_1+3.333\\ r_2\\ \hline 13.333\\ 153.333\end{array}\right\}$$

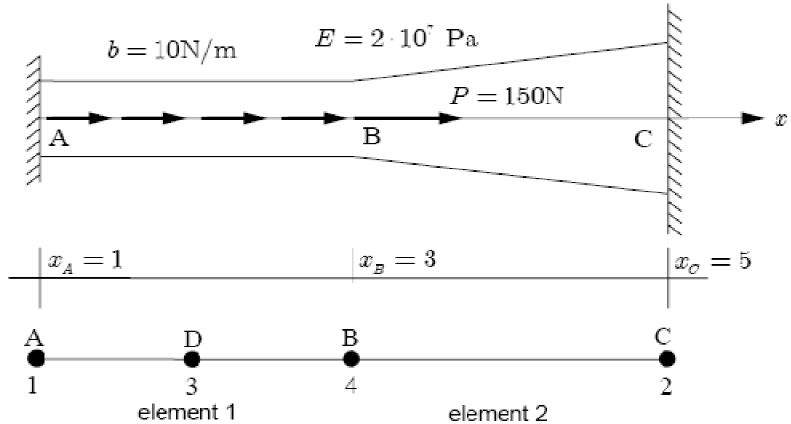
Solution step



$$10^6 \begin{bmatrix} 5.333 & -2.6667 \\ -2.6667 & 3.8333 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 13.333 \\ 153.333 \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 34.5 \\ 64 \end{Bmatrix} 10^{-6} m$$

$$[K] = 10^6 \begin{bmatrix} -2.6667 & 0.333 \\ 0 & -1.5 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} r_1 + 3.333 \\ r_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} -74.02 \\ -96 \end{Bmatrix} Nt$$

Displacement calculation

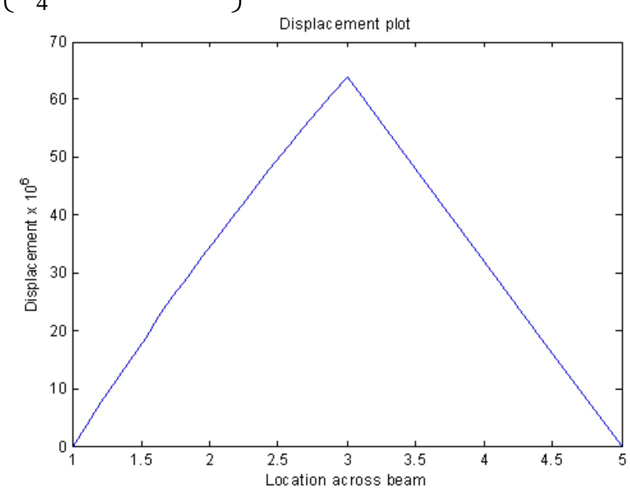


$$u^1(x) = \left[N^1\right]\{d^1\} = \left[\frac{1}{2}(x-2)(x-3) \quad -(x-1)(x-3) \quad \frac{1}{2}(x-1)(x-2)\right]\begin{Bmatrix} u_1 = 0 \\ u_3 = 34.5\,10^{-6} \\ u_4 = 64\,10^{-6} \end{Bmatrix} \Rightarrow$$

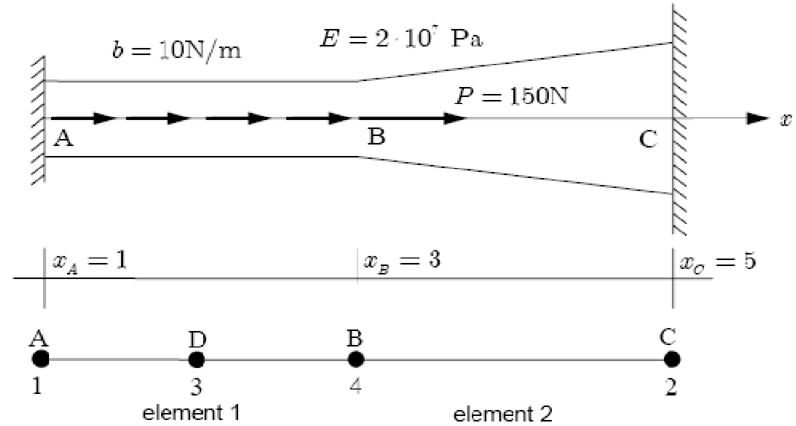
$$u^1(x) = \left[N^1\right]\{d^1\} = (-2.5x^2 + 42x - 39.5)10^{-6}\,m$$

$$u^2(x) = \left[N^2\right]\{d^2\} = \left[\frac{1}{2}(5-x) \quad \frac{1}{2}(x-3)\right]\begin{Bmatrix} u_4 = 64\,10^{-6} \\ u_2 = 0 \end{Bmatrix} \Rightarrow$$

$$u^2(x) = (160 - 32x)10^{-6}\,m$$



Post processing: Stress calculation

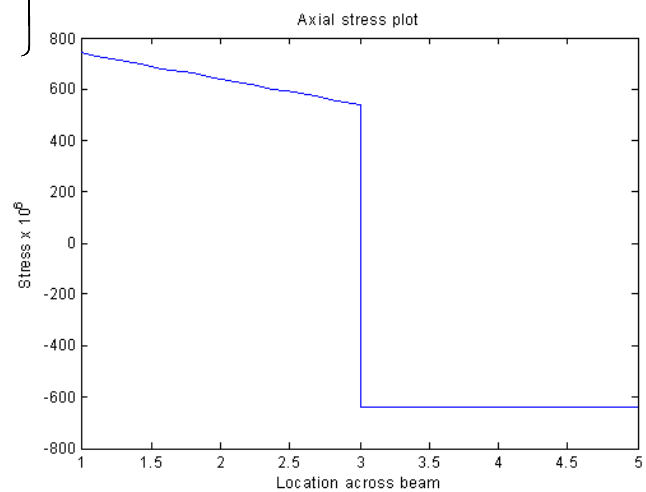


$$\sigma^1(x) = E\left[B^1\right]\{d^1\} = \left[(x-\frac{5}{2}) \quad 4-2x \quad (x-\frac{3}{2})\right]\begin{Bmatrix} u_1 = 0 \\ u_3 = 34.510^{-6} \\ u_4 = 6410^{-6} \end{Bmatrix} \Rightarrow$$

$$\sigma^1(x) = 840 - 100x \;\; Pa$$

$$\sigma^2(x) = E\left[B^2\right]\{d^2\} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_4 = 6410^{-6} \\ u_2 = 0 \end{Bmatrix} \Rightarrow$$

$$\sigma^2(x) = -640 \; Pa$$



Example problem

- We revisit a boundary value problem with generalized boundary conditions (all coefficients shown are non zero).

$$-\frac{d}{dx}(k(x)\frac{du}{dx}) + q(x)u(x) = f(x), x \in (0,l)$$

$$\alpha_0 \frac{du}{dx}(0) + \beta_0 u(0) = \gamma_0$$

$$\alpha_l \frac{du}{dx}(l) + \beta_l u(l) = \gamma_l$$

- We here start with weak formulations written separately for each element.
- This process is identical with what was done before but it is helpful as it shows how to include discontinuities in fluxes within the domain.
- Here we use index rather than matrix notation (i.e. instead of writing Ax=b, we write $\sum_{j=1}^{N} A_{ij} u_j = F_i, i = 1, 2, ..., N$).

Example problem: Weak form

- For each element *e*, we can write the following weak form:

$$\int_{x_1^e}^{x_2^e} k(x)\frac{du_h^e}{dx}\frac{dw_h^e}{dx}dx \underbrace{-k(x)\frac{du_h^e}{dx}}_{\substack{\sigma_h^e \\ Flux}} w_h^e \Big|_{x_1^e}^{x_2^e} + \int_{x_1^e}^{x_2^e} q(x)u_h^e(x)w_h^e(x)dx = \int_{x_1^e}^{x_2^e} f(x)w_h^e(x)dx, \forall w_h^e \in H^1(\Omega_e) \Rightarrow$$

$$\int_{x_1^e}^{x_2^e} k(x)\frac{du_h^e}{dx}\frac{dw_h^e}{dx}dx + \int_{x_1^e}^{x_2^e} q(x)u_h^e(x)w_h^e(x)dx = \int_{x_1^e}^{x_2^e} f(x)w_h^e(x)dx$$

$$+\sigma_h^e(x_1^e)w_h^e(x_1^e) - \sigma_h^e(x_2^e)w_h^e(x_2^e), \forall w_h^e \in H^1(\Omega_e)$$

- Let us clarify the notation once more:
  - We use for both *u* and *w* the indices $\cdot_h^e$ to emphasize that we work with their restrictions on element *e*.
  - Note that since we write the weak form for element *e* alone, we now have (unknown) flux terms appearing at the 2 ends of each element.

Example problem: Weak form

$$\int_{x_1^e}^{x_2^e} k(x)\frac{du_h^e}{dx}\frac{dw_h^e}{dx}dx + \int_{x_1^e}^{x_2^e} q(x)u_h^e(x)w_h^e(x)dx = \int_{x_1^e}^{x_2^e} f(x)w_h^e(x)dx$$

$$+\sigma_h^e(x_1^e)w_h^e(x_1^e) - \sigma_h^e(x_2^e)w_h^e(x_2^e), \forall w_h^e \in H^1(\Omega_e)$$

- For each element e, we approximate $u_h^e(x) = \sum_{j=1}^{N_e} u_h^e(x_j^e)N_j^e(x)$ and taking $w_h^e(x) = N_i^e, i = 1,..,N_e$ results in the following discretized equations:

$$\sum_{j=1}^{N_e}\left(\int_{x_1^e}^{x_2^e} k(x)\frac{dN_j^e}{dx}\frac{dN_i^e}{dx}dx\right)u_h^e(x_j^e) + \sum_{j=1}^{N_e}\left(\int_{x_1^e}^{x_2^e} q(x)N_j^e(x)N_i^e(x)dx\right)u_h^e(x_j^e) = \int_{x_1^e}^{x_2^e} f(x)N_i^e(x)dx$$

$$+\sigma_h^e(x_1^e)N_i^e(x_1^e) - \sigma_h^e(x_2^e)N_i^e(x_2^e), i = 1,\ldots,N_e$$

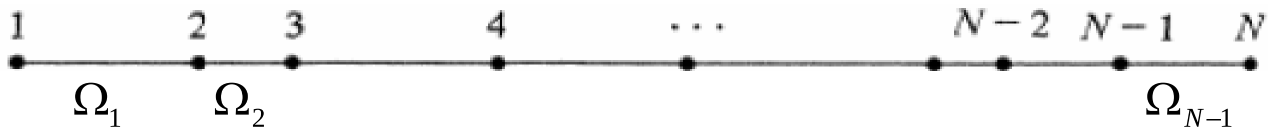
- Thus as we have seen before, the element stiffness matrix and load vector are:

$$K_{ij}^e = \int_{x_1^e}^{x_2^e} k(x)\frac{dN_j^e}{dx}\frac{dN_i^e}{dx}dx + \int_{x_1^e}^{x_2^e} q(x)N_j^e(x)N_i^e(x)dx$$

$$f_i^e = \int_{x_1^e}^{x_2^e} f(x)N_i^e(x)dx, i = 1,\ldots,N_e$$

Example problem: Element equations

$$\sum_{j=1}^{N_e} K_{ij}^e u_h^e(x_j^e) = f_i^e + \sigma_h^e(x_1^e) N_i^e(x_1^e) - \sigma_h^e(x_2^e) N_i^e(x_2^e), i = 1, \ldots, N_e$$



- Let us write these equations explicitly for a linear element.

$$K_{11}^e u_h^e(x_1^e) + K_{12}^e u_h^e(x_2^e) = f_1^e + \sigma_h^e(x_1^e)$$
$$K_{21}^e u_h^e(x_1^e) + K_{22}^e u_h^e(x_2^e) = f_2^e - \sigma_h^e(x_2^e)$$

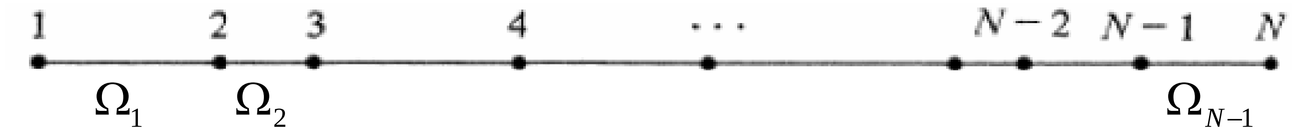
- For the first element, these equations are:

$$K_{11}^1 u_1 + K_{12}^1 u_2 = f_1^1 + \sigma(0)$$
$$K_{21}^1 u_1 + K_{22}^1 u_2 = f_2^1 - \sigma(x_2^-)$$

where $\sigma(x_2^-)$ is the heat flux at node 2 approached from the left.

Example problem: Assembly process

$$\sum_{j=1}^{N_e} K_{ij}^e u_h^e(x_j^e) = f_i^e + \sigma_h^e(x_1^e) N_i^e(x_1^e) - \sigma_h^e(x_2^e) N_i^e(x_2^e), i = 1, \ldots, N_e$$



- 1st element

$$K_{11}^1 u_1 + K_{12}^1 u_2 = f_1^1 + \sigma(0)$$
$$K_{21}^1 u_1 + K_{22}^1 u_2 = f_2^1 - \sigma(x_2^-)$$

- 2nd element

$$K_{11}^2 u_2 + K_{12}^2 u_3 = f_1^2 + \sigma(x_2^+)$$
$$K_{21}^2 u_2 + K_{22}^2 u_3 = f_2^2 - \sigma(x_3^-)$$

- Assembling the elements equations finally gives:

$$K_{11}^1 u_1 + K_{12}^1 u_2 = f_1^1 + \sigma(0)$$
$$K_{21}^1 u_1 + (K_{22}^1 + K_{11}^2) u_2 + K_{12}^2 u_3 = f_2^1 + f_1^2 + \underbrace{\sigma(x_2^+) - \sigma(x_2^-)}_{\substack{[\![\sigma(x_2)]\!] \\ \text{flux disctontinuity} \\ \text{at } x_2}}$$
$$K_{21}^2 u_2 + \left(K_{22}^2 + K_{11}^3\right) u_3 + K_{12}^3 u_4 = f_2^2 + f_1^3 + [\![\sigma(x_3)]\!]$$
$$\ldots$$
$$\ldots$$
$$K_{21}^{N-1} u_1 + K_{22}^{N-1} u_2 = f_2^{N-1} - \sigma(l)$$

Example problem: Discretized equations

$$K_{11}^{1}u_1 + K_{12}^{1}u_2 = f_1^{\,1} + \sigma(0)$$

$$K_{21}^{1}u_1 + (K_{22}^{1} + K_{11}^{2})u_2 + K_{12}^{2}u_3 = f_2^{\,1} + f_1^{\,2} + [\![\sigma(x_2)]\!]$$

$$K_{21}^{2}u_2 + \left(K_{22}^{2} + K_{11}^{3}\right)u_3 + K_{12}^{3}u_4 = f_2^{\,2} + f_1^{\,3} + [\![\sigma(x_3)]\!]$$

…

…

$$K_{21}^{N-2}u_{N-2} + \left(K_{22}^{N-2} + K_{11}^{N-1}\right)u_{N-1} + K_{12}^{N-1}u_N = f_2^{\,N-2} + f_1^{\,N-1} + [\![\sigma(x_{N-1})]\!]$$

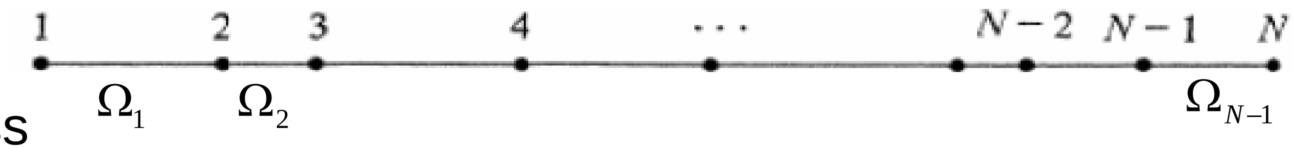
$$K_{21}^{N-1}u_1 + K_{22}^{N-1}u_2 = f_2^{\,N-1} - \sigma(l)$$

You can now account for flux discontinuities within the domain! For example let

$$[\![\sigma(x_3)]\!] = \widehat{f}_3$$

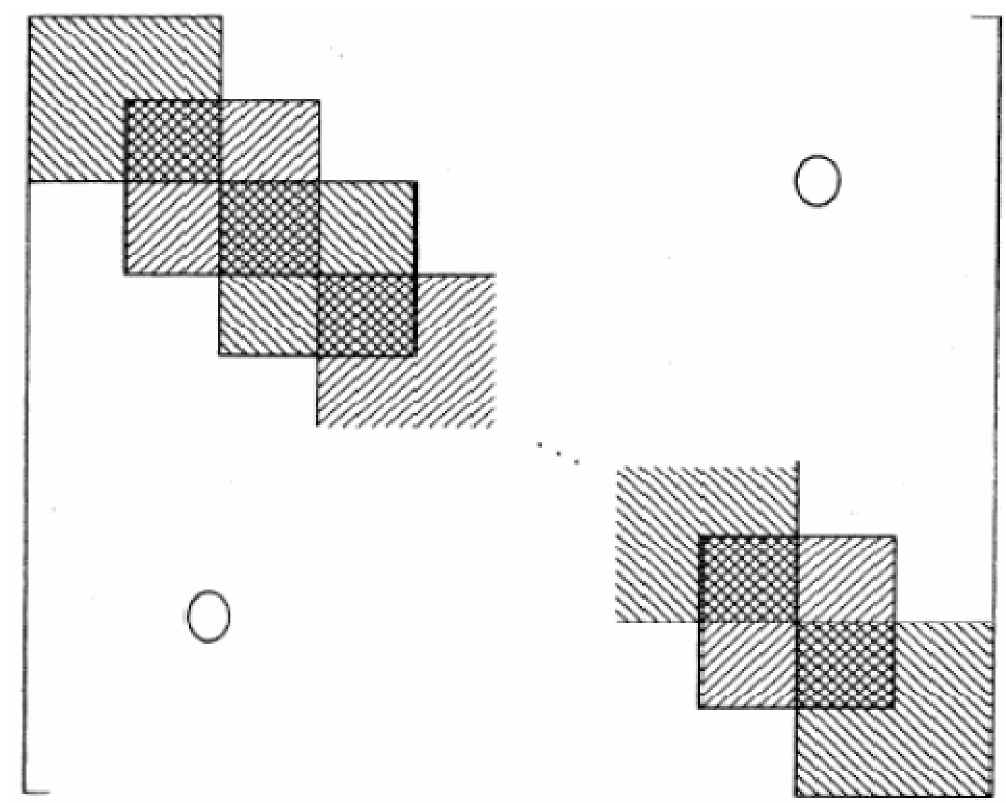
Example problem:Assembly process

- A finite-element mesh with *N* nodes and *N - 1* two-node linear elements and the assembled stiffness matrix.



- The shaded blocks of entries representing the contributions of each element.

- The symbols ***0*** represent the fact that outside the diagonal blocks all entries are zero.



Example problem: Boundary conditions

$$
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & & & & \\
k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & & & \\
 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & & \\
 & & k_{21}^3 & k_{22}^3 + k_{11}^4 & & \\
 & & & & & \\
 & & & & k_{22}^{N-2} + k_{11}^{N-1} & k_{12}^{N-1} \\
 & & & & k_{21}^{N-1} & k_{22}^{N-1}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ u_{N-1} \\ u_N \end{Bmatrix}
=
\begin{Bmatrix} f_1^1 + \sigma(0) \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 + \hat{f} \\ . \\ . \\ f_2^{N-2} + f_1^{N-1} \\ f_2^{N-1} - \sigma(l) \end{Bmatrix}
$$

- Finally, recall our 2 generalized boundary conditions:

$$\alpha_0 \frac{du}{dx}(0) + \beta_0 u(0) = \gamma_0 \Rightarrow \quad \sigma(0) = -k(0)\frac{du}{dx}(0) = -k(0)\left(\frac{\gamma_0}{\alpha_0} - \frac{\beta_0}{\alpha_0}u(0)\right)$$

$$\alpha_l \frac{du}{dx}(l) + \beta_l u(l) = \gamma_l \Rightarrow \quad \sigma(l) = -k(l)\frac{du}{dx}(l) = -k(l)\left(\frac{\gamma_l}{\alpha_l} - \frac{\beta_l}{\alpha_l}u(l)\right)$$

Example problem: Final linear system of equations

$$\begin{bmatrix} k_{11}^1 - k(0)\dfrac{\beta_0}{\alpha_0} & k_{12}^1 & & & & \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & & & \\ & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & & \\ & & k_{21}^3 & k_{22}^3 + k_{11}^4 & & \\ & & & & k_{22}^{N-2} + k_{11}^{N-1} & k_{12}^{N-1} \\ & & & & k_{21}^{N-1} & k_{22}^{N-1} + k(l)\dfrac{\beta_l}{\alpha_l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ u_{N-1} \\ u_N \end{Bmatrix} = \begin{Bmatrix} f_1^1 - k(0)\dfrac{\gamma_0}{\alpha_0} \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 + \widehat{f} \\ . \\ . \\ f_2^{N-2} + f_1^{N-1} \\ f_2^{N-1} + k(l)\dfrac{\gamma_l}{\alpha_l} \end{Bmatrix}$$

- Once you compute the nodal solution, the FEM solution in each element is:

$$u_h^e(x) = \sum_{j=1}^{N_e} u_h^e(x_j^e) N_j^e(x), \qquad \sigma_h^e(x) \equiv -k(x)\frac{du_h^e(x)}{dx} = -k(x)\left(\sum_{j=1}^{N_e} u_h^e(x_j^e)\frac{dN_j^e(x)}{dx}\right).$$