Due Monday, November 2nd, 12:00 midnight

Problem 1 – Axisymmetric analysis (MatLab)

The problem of stress analysis of solids of revolution (axisymmetric solids) under axisymmetric loads is of considerable practical interest. This problem is similar to those of plane stress since the displacements are confined to only two directions (radial and axial).

For axisymmetric formulation, the elasticity equation must be expressed in terms of cylindrical coordinates. When the elasticity problem degenerates from three-dimensions to axisymmetry, two shearing stress components vanish. These vanishing components due to symmetry are $\tau_{r\theta}$ and $\tau_{z\theta}$ in the $r\theta z$ coordinates system where r is the radial direction, θ is the circumferential direction, and z is the axial direction. Hence, the remaining stress components are $\boldsymbol{\sigma} = \left(\sigma_r\ \sigma_\theta\ \sigma_z\ \tau_{rz}\right)^T$ Similarly, the remaining strains are $\boldsymbol{\varepsilon} = \left(\varepsilon_r\ \varepsilon_\theta\ \varepsilon_z\ \gamma_{rz}\right)^T$.

The material property matrix $\mathbf{D}$ in $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ for the axisymmetric problem is

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

The kinematic equations relating strains and displacements are

$$\begin{pmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{pmatrix}$$

For an axisymmetric problem, since all computations reduce to those in the $r-z$ plane, the finite element model consists of a discretization of this plane area. The element equations are developed using the same interpolation functions as those used for a plane stress problem.

(a) Using 4-node quadrilateral element, substitute the shape functions into the kinematic equation and derive the $\mathbf{B}$ matrix.
The element stiffness matrix can be expressed as

$$\mathbf{K}^e = \int_z\int_r\int_\theta \mathbf{B}^T\mathbf{D}\mathbf{B}\,d\theta\,dr\,dz = 2\pi\int_r\int_z r\mathbf{B}^T\mathbf{D}\mathbf{B}\,dr\,dz$$

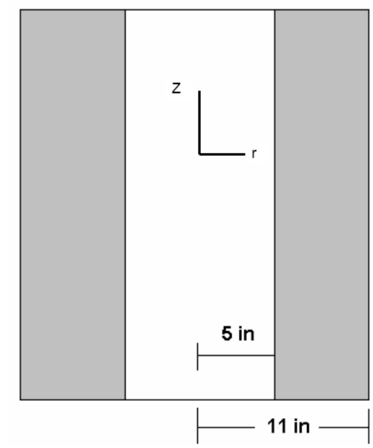
The equivalent load vector due to distributed loads:

$$f^e = \int_s N_c \begin{pmatrix} q_r \\ q_z \end{pmatrix} 2\pi r ds$$

where N_c are the displacement interpolation functions on the boundary.

Modify the MatLab code to solve this axisymmetric problem.

(b) To illustrate this type of analysis consider the problem of finding the stresses in a thick open-ended cylinder with an internal pressure (such as a pipe discharging to the atmosphere). The steel cylinder below has an inner radius of 5 inches and an outer radius of 11 inches.



In both drawings the length of the object is arbitrary and represents a segment of a long, open-ended cylinder. The z axis is the axis of symmetry. The cylinder can be generated by revolving a rectangle 6 inches wide and of arbitrary height 360 degrees about the z axis.

Since the height of the segment considered s arbitrary, we will use a segment 1 inch in height for the finite element model. The geometry is shown below

Solve the problem using your modified MatLab code. A pressure of 1000 psi is applied at the left boundary. The cylinder is constrained on the bottom line in the axial direction $(v = 0)$. This simply prevents rigid body motion in the z direction. No other displacement boundary conditions are required. The radial movement is prevented by the 'hoop' tension in the cylinder. Young's modulus $E = 3\times10^7$ Pa and Poisson's ratio $\nu = 0.3$.

Check the deformed shape to see if it's reasonable. Compare your solutions with the theoretic results: at the inside of the cylinder $\sigma_r = -1000$ psi and

$\sigma_\theta = 1521$psi, radial displacement $u_r = 0.305\times10^{-3}$inches. At the outside of the cylinder $\sigma_r = 0$psi and $\sigma_\theta = 520$ psi, radial displacement $u_r = 0.191\times10^{-3}$ inches.

Problem 2 – Analysis of cracks for inplane deformation of a thin plate (MatLab)
Consider the plate shown following subjected to the tensile stress σ_0 with a crack as shown.

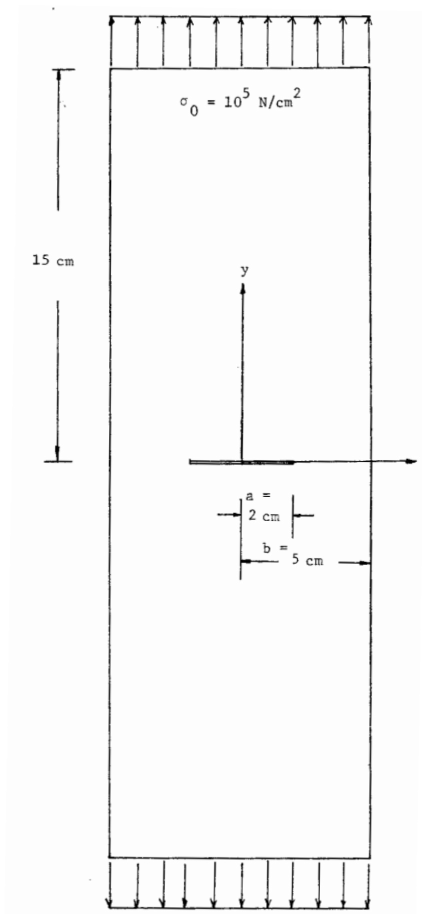


Figure 7: Stress analysis near crack tips: Young's modulus $E = 2.1\times10^7$ Nt/cm^2, Poisson's ratio $v = 0.29$, $\sigma_0 = 10^5$ N/cm^2

Using symmetry, analyze only one quarter of the plate using quadrilateral elements. To simulate the crack using the finite element method, we assume the portion of the crack face on the boundary is traction free.

1. Plot the deformed body
2. The stress distribution along the x axis at $\theta = 0$ (σ_{yy}) together with the analytical solution given as:

$$\sigma_{xx} = \frac{K_I}{(2\pi r)^{1/2}} \cos \tfrac{1}{2}\theta (1 - \sin \tfrac{1}{2}\theta \sin \tfrac{3}{2}\theta)$$

$$\sigma_{yy} = \frac{K_I}{(2\pi r)^{1/2}} \cos \tfrac{1}{2}\theta (1 + \sin \tfrac{1}{2}\theta \sin \tfrac{3}{2}\theta)$$

$$\sigma_{xy} = \frac{K_I}{(2\pi r)^{1/2}} \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta \cos \tfrac{3}{2}\theta$$

where $K_I = \sigma_0 \sqrt{\pi a}$, σ_0 is the stress σ_{yy} at ∞ and a is the half size of the crack.. The origin of the polar coordinates defined above is taken at the crack tip $(2,0)$ Here the analytical solutions corresponds to an infinite plate with a crack the length of which is $2a$. K_I is known as the stress intensity factor of the first mode in fracture mechanics and depends on the shape of the domain and the applied load. For a domain of finite width as in our problem, Lawn and Wilshaw (1975) reported that

$$(K_I)_a = \sigma_0 \sqrt{\pi a} \sqrt{\frac{2b}{\pi a} \tan \frac{\pi a}{2b}}$$

Plot the stress distribution σ_{yy} along x axis at $\theta = 0^\circ$ from $r = 1\,(x = 2)$ to 3 $(x = 5)$ with the analytical solution. Also plot the stress distribution σ_{yy} from x = 0 to 5 to examine the rapid growth of the stress near the crack tip.

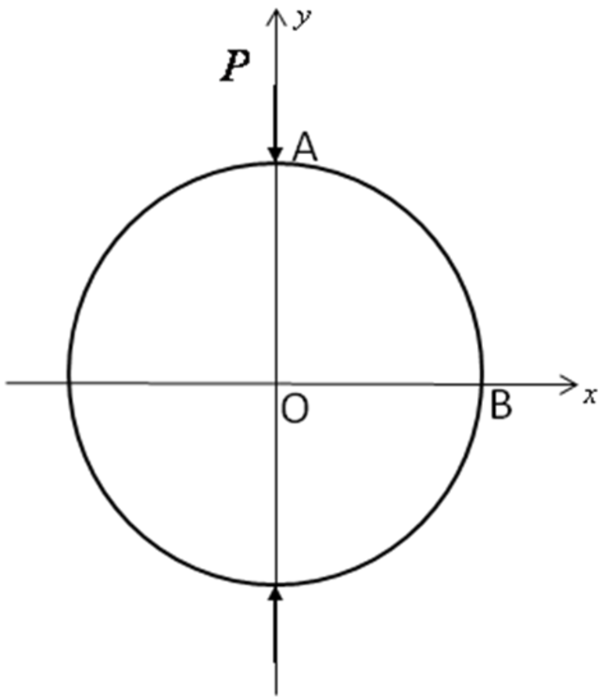
3. Compute and plot the maximum shear stress $\tau_{\max}$.
4. There are several methods to compute K_I using the finite element method. For example using the stress solution near the crack tips yields:

$$K_I = \sqrt{2\pi r}\,\sigma_{yy} \Big/ \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

By finding σ_{yy} along the x axis corresponding to $\theta = 0$ using finite element solutions, evaluate the value of K_I and the error vs. for example the solution reported bv Lawn and Wilshaw.

Important note: In order to trace the rapid growth of the stress near the crack tip, very small size finite elements should be allocated there.

Problem 3 – Circular disk subjected to point loading (MatLab)



Consider a circular disk loaded by two concentrated forces of $P = 10\,\text{N}$ each directed along a diagonal. The material of the disk is assumed to be linearly elastic. Furthermore, for simplicity we consider the loads to be slowly applied so that inertial effects may be ignored. Thus, the model to be solved is a simple plane stress problem. Since the loading is symmetric and we assume the material to be isotropic, it is only necessary to construct a mesh for one quadrant ABO of the circular disk. $E = 1000\,\text{N/m}^2, v = 0.25$. The boundary of the disk is free from external forces. The diameter of the disk $d = 2\,\text{m}$.

The stress component σ_{yy} on horizontal plane is given analytically as (Timoshenko and Goodier, *1970)*

$$\sigma_{yy} = \frac{2P}{\pi d}\left[1 - \frac{4d^4}{\left(d^2 + 4x^2\right)^2}\right]$$

1. Plot the deformed shape.
2. Plot the displacement and stress field.
3. Plot σ_{yy} along OB and compare the finite element solution with the analytical solution.