Due Monday, October 26th, 12:00 midnight

Problem 1 – Thermal expansion (hand calculation)
An elastic body subjected to temperature tends to expansion. This expansion is given in the form of thermal (prescribed) strain as

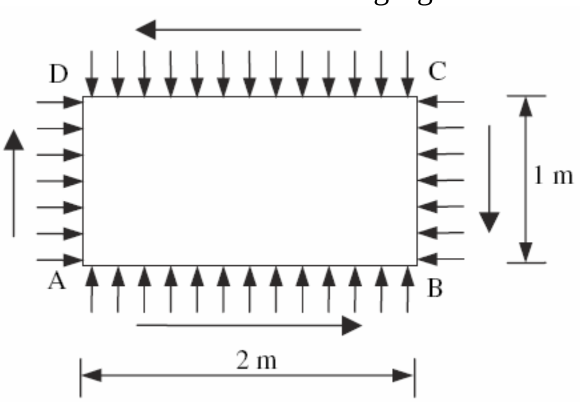
$$\boldsymbol{\varepsilon}^0 = \alpha T[1\ 1\ 0]^T,$$

where α is the thermal expansion coefficient (for isotropic materials) and T is the temperature. The stress is

$$\boldsymbol{\sigma} = D(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0)$$

where the term corresponding to thermal strains is prescribed temperature data. Develop a weak form and finite element matrices for the case of loads arising due to thermal expansion. Hint: Substitute the above into the weak form derived in class and repeat the derivation shown there.

Problem 2 – Plane stress (hand calculation)
Consider a rectangular panel as shown in the following figure:

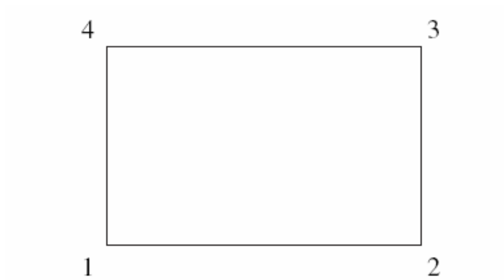


The panel is modeled using a plane stress linear elastic material with the following properties: Young's modulus $E = 3\times10^{11}$ Pa and Poisson's ratio $\nu = 0.3$. The essential boundary conditions are

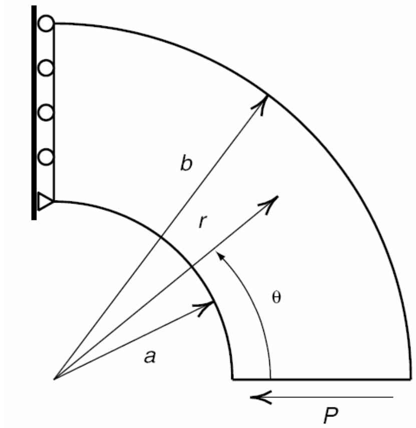
$$u_{Ax} = u_{Ay} = u_{By} = 0$$

The natural boundary conditions are as follows. Along each edge of the panel, the prescribed traction consists of normal and lateral components, both equal to 10^3 N m^{-1} as shown in the figure.

Discretize the panel using a single rectangular element as shown below. For convenience, use identical global and local numbering. Calculate nodal displacements and stresses at the element Gauss points.



Problem 3 – Circular beam subjected to end shear (MatLab)



We consider a circular beam in a state of plane stress. The solution to the problem is given in Timoshenko and Goodier based on the use of a stress function. The geometry and loading for the problem are shown in the above figure. The solution for stresses is given by

$$\sigma_{rr} = \frac{P}{N}\left[r + \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right]\sin\theta$$

$$\sigma_{\theta\theta} = \frac{P}{N}\left[3r - \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right]\sin\theta$$

$$\tau_{r\theta} = -\frac{P}{N}\left[r + \frac{a^2 b^2}{r^3} - \frac{a^2 + b^2}{r} \right]\cos\theta$$

where $N = a^2 - b^2 + (a^2 + b^2)\log b/a$. For the restraints shown in Figure the solution for displacements is given by

$$u_r = \frac{P}{NE}\left\{ \left[\frac{1}{2}(1-3\nu)r^2 - \frac{a^2 b^2 (1+\nu)}{2r^2} - (a^2+b^2)(1-\nu)\log r \right]\sin\theta + (a^2+b^2)(2\theta-\pi)\cos\theta \right\}$$
$$-K\sin\theta$$

$$u_\vartheta = \frac{P}{NE}\left\{ \left[\frac{1}{2}(5+\nu)r^2 - \frac{a^2 b^2 (1+\nu)}{2r^2} + (a^2+b^2)\left[(1-\nu)\log r + (1+\nu)\right] \right]\left(-\cos\theta\right) - (a^2+b^2)(2\theta-\pi)\sin\theta \right\}$$
$$-K\cos\theta$$

Note for $u_r(a,\pi/2) = 0$, we obtain

$$K = \frac{P}{NE}\left[\frac{1}{2}(1-3\nu)a^2 - \frac{b^2(1+\nu)}{2} - (a^2+b^2)(1-\nu)\log a \right]$$

In the above E and ν are the elastic modulus and Poisson ratio; a and b are inner and outer radii, respectively.

For this solution the displacement u_r for $\theta = 0$ is constant and given by

$$u_r(r,0) = -\frac{\pi P}{EN}(a^2+b^2) = u_0$$

Thus, instead of computing the nodal forces for the traction on this boundary, we merely set all the nodal displacements in the x direction to a constant value.

For the numerical solution we choose the properties

$$a = 5;\ b = 10;\ \text{thickness } t = 1;\ u_0 = -0.01;\ E = 10000 \text{ and } \nu = 0.25$$

In addition the displacements on the boundary

$$u(x,0) = u_0 \text{ and } u(0,y) = v(0,a) = 0$$

Solve the problem by modifying the MATLAB code.

(a) Plot the deformed structure;

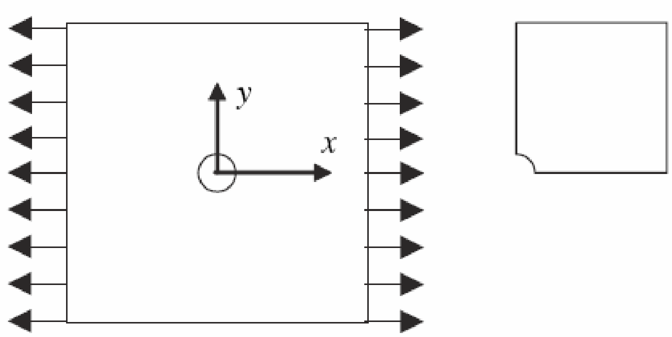
(b) Compute the elastic energy $\mathbf{d}^T\mathbf{K}\mathbf{d}$, where d is your finite element solution and K is the global stiffness matrix. Compare your result with the exact value $\frac{1}{\pi}\left[\log 2 - 0.6\right]$.

(c) Plot the contour of $\sigma_{rr}, \sigma_{\theta\theta}$ and $\sigma_{r\theta}$. Compare your result with finite element solution.

(d) Compare the u_r and u_θ displacements from the finite element solution to the exact values at $r = a$.

Note: The finite element solution from the Matlab Code is in Cartesian coordinates while the analytical solutions are given in Polar coordinates. Therefore, you need to transform the result to the same coordinate in order to make the comparison.

Problem 4 – Thin plate with a hole in tension (MatLab)



Consider a tension problem involving a thin linearly elastic plate with a hole as shown in the figure. Suppose that the plate is a homogeneous isotropic elastic body.

The plate is of unit thickness and subject to tension in the horizontal direction. Because of symmetry in the model and loading, model only one quarter of the plate. Use ANSYS to generate the gird files using 4-node quadrilateral elements.

The plate is $20\text{ cm} \times 20\text{ cm}$ and the radius of the hole is 2.5 cm. Assume Young's modulus is $2.1\times10^7\text{ N/cm}^2$ and Poisson's ration is 0.29. The uniform load applied is $\sigma_0 = 100\text{ N/cm}$.

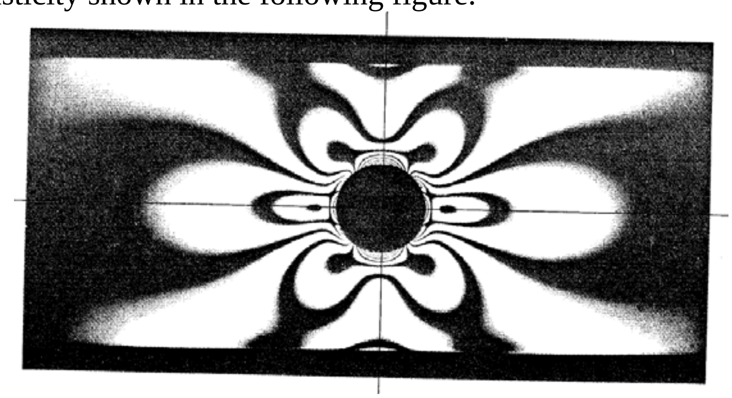
To check the quality of the finite element solution, compare your solution with that of an infinite plate. For a circular hole in an infinite plate subjected to a pure uniaxial tension or compression σ_0, at ∞, there exists an exact solution for the stresses (Timoshenko and Goodier, 1970):

$$\sigma_{rr} = \frac{\sigma_0}{2}\left\{\left[1-\left(\frac{a}{r}\right)^2\right]+\left[1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta\right\}$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left\{\left[1+\left(\frac{a}{r}\right)^2\right]-\left[1+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta\right\}$$

$$\sigma_{r\theta} = -\frac{\sigma_0}{2}\left[1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$

a. Provide a plot showing the FE and exact σ_{xx} stress distribution along the y -axis. What is the stress concentration factor?
b. Show the deformed configuration
c. Compute and plot the distribution of the finite element principal stress solution (σ_1, σ_2) and the maximum shear stress solution.
d. Note that an interesting comparison can be made with the stress pattern obtained by photoelasticity in which fringes correspond to the maximum shear stress $\tau_{max} = 1/2(\sigma_1 - \sigma_2)$. Verify that the contour lines of τ_{max} obtained by the finite element approximation are quite similar to the fringe pattern obtained by the photoelasticity shown in the following figure.



Problem 5 – Thin plate with a hole in tension (ANSYS)

Repeat problem 4 using Ansys and compare the results with those obtained using MatLab for the same discretization.