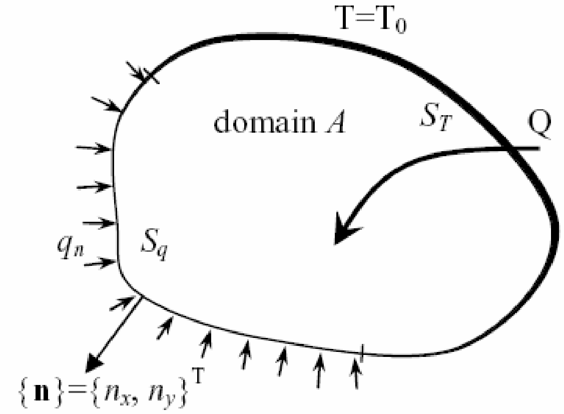
Due Wednesday, October 14th, 12:00 midnight

Problem 1 – Weak form of the steady-state heat flow problem in two-dimensions (hand calculation)

Consider the problem definition in the following figure:



The governing differential equation is

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right)+Q=0$$

with Q a known heat source per unit area.

Note that the compact form for this partial differential equation is: $\nabla \bullet \left([k]\nabla T\right)+Q=0$

where the conductivity matrix is $[k]=\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$.

Consider the following boundary conditions:

$$\begin{cases} T = T_0 \text{(prescribed)} \quad \text{on} \quad S_T \\ \quad q_n = q_n \quad \text{on} \quad S_q \end{cases}$$

(a) Multiply the above differential equation by virtual temperature $w=\delta T$ and integrate over the domain A. Recall that $w=\delta T$ is zero at the boundary where the temperature is prescribed.
Integrate by parts each term separately in the differential equation using the following:

$$\iint_A \left(\frac{\partial}{\partial x}\left(k_x \frac{\partial T_x}{\partial x}\right) w + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) w \right) dA = \int_S \left(k_x \frac{\partial T_x}{\partial x} n_x w + k_y \frac{\partial T_y}{\partial y} n_y w \right) dS - \iint_A \left(\frac{\partial w}{\partial x} k_x \frac{\partial T_x}{\partial x} + \frac{\partial w}{\partial y} k_y \frac{\partial T_y}{\partial y} \right) dA$$

Show that this is the equivalent component form of the compact expression used in class and also given in the lecture notes:

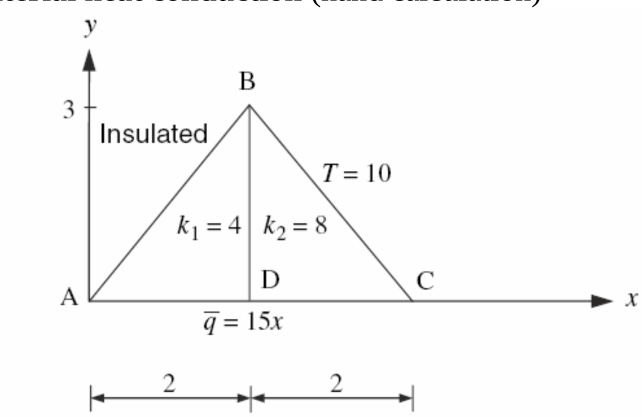
$$\iint_A \nabla \bullet \left([k]\nabla T \right) w dA = \int_S \underbrace{[k]\nabla T \bullet n}_{q_n} w dS - \iint_A \nabla w \bullet [k]\nabla T dA$$

where $q_n = q_x n_x + q_y n_y$ is the normal heat flux on the surface whose unit normal vector is $\vec{\mathbf{n}} = n_x \vec{i} + n_y \vec{j}$. Provide the final weak form for this problem.

(b) Consider now a convection boundary condition given by $q_n = h(T_f - T) \; on \; S_q$. For this case, provide the new weak form and then derive the finite element matrix equation $\mathbf{K}^e \mathbf{d}^e = \mathbf{f}^e$ of the heat flow problem. You don't need to do any specific element calculations. Leave the element stiffness and load vectors in terms of integrals involving the basis functions $\left[N^e \right]$ and their derivatives $\left[B^e \right]$.

Show the contributions to the stiffness and load vectors that result from the convection boundary condition.

Problem 2 – Bi-material heat conduction (hand calculation)

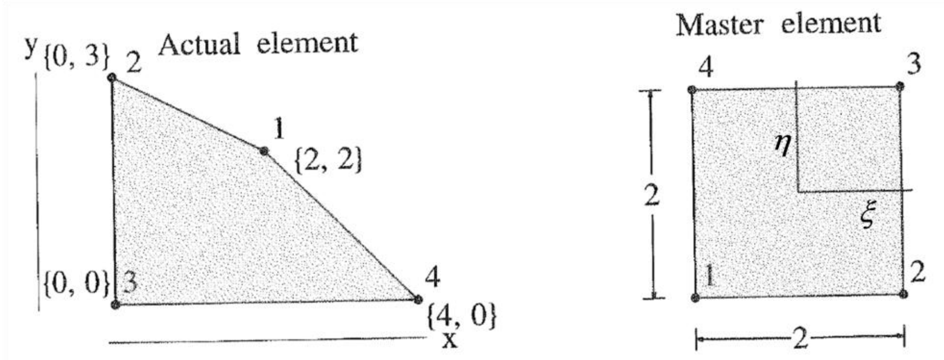


Consider a triangular panel made of two isotropic materials with thermal conductivities of $k_1 = 4 \text{ W } ^\circ C^{-1}$ and $k_2 = 8 \text{ W } ^\circ C^{-1}$ as shown in the figure. A constant temperature of $\bar{T} = 10^\circ C$ is prescribed along the edge BC. The edge AB is insulated and a linear distribution of flux, $\bar{q} = 15x \text{ W m}^{-1}$, is applied along the edge AC. Point source $P = 45$ W is applied at $(x = 3, y = 0)$. Plate dimensions are in meters.

For the finite element mesh, consider two triangular elements, ABD and BDC. Carry out calculations manually and find the temperature and flux distributions in the plate.

Problem 3 – Mapping Quadrilaterals (hand calculation)

A four-node quadrilateral element is shown in the figure below



(a) Develop an appropriate mapping to map the actual element into the 2×2 master element shown in the figure.
(b) Verify the mapping is good.
(c) Compute $\partial N_3 / \partial y$, where $N_3(\xi,\eta)$ is the third interpolation function for the master element. Give the value of the derivative at node 3 of the element.
(d) Compute $\iint_A 64 N_2 N_3 dA$, where $N_2(\xi,\eta)$ and $N_3(\xi,\eta)$ are the second and the third interpolation functions for the master element and A is the area of the actual element. Use numerical integration with a 1×1 Gauss quadrature formula.
(e) Compute $\iint_C 4 N_{3c} dc$, where c is the line 3-4 of the element and N_{3c} is the third interpolation function for the master element expressed in a coordinate that runs along this line. Use numerical integration with a two-point Gauss quadrature formula.

Problem 4 – Convection heat transfer (MatLab)

Here we consider the FEM analysis of heat transfer problems that involve convection phenomena (i.e. energy transfer between a solid body and a surrounding fluid medium). The governing equation in a special case considered here is taken as

$$-\frac{\partial}{\partial x}(k_x \frac{\partial T}{\partial x}) - \frac{\partial}{\partial y}(k_y \frac{\partial T}{\partial y}) = f \text{ in } \Omega$$

where T is the temperature (in $^\circ$C), k_x and k_y are the conductivities [in W/(m$\cdot^\circ$C)] along the x and y directions, respectively and f is the internal heat generation per unit volume in (W/m^3).

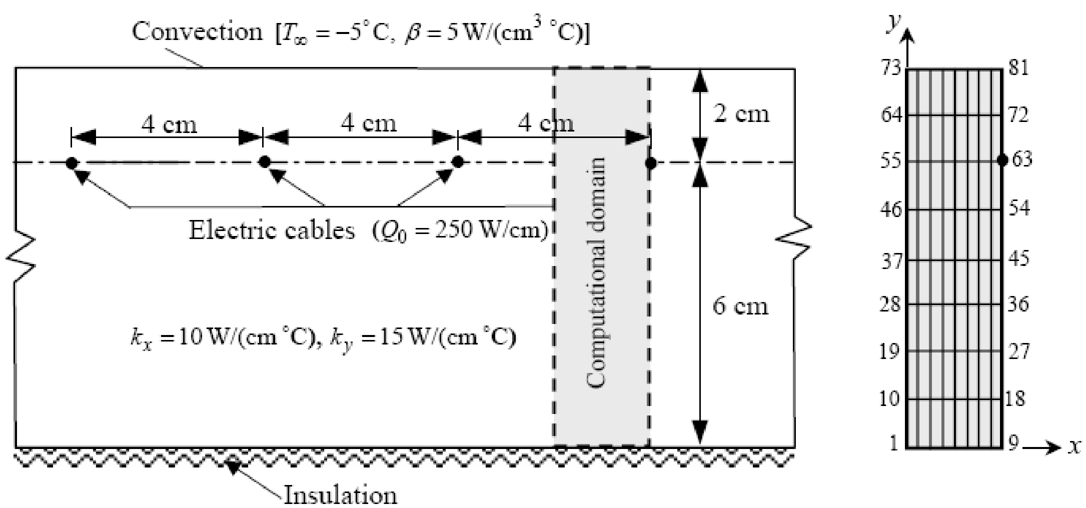
The essential boundary conditions here involve the specification of the temperature T. The natural boundary condition involves the specification of the heat flux $\hat{q}_n$ such that:

$$(k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y) + \beta(T - T_\infty) - \hat{q}_n = 0$$

where β is the convective heat transfer coefficient [in W/(m·°C)] and T_∞ is (ambient) temperature of the surrounding fluid medium and $\hat{q}_n$ is specified heat flux.. The first term accounts for heat transfer by conduction, the second by convection and the third accounts for the specified heat flux.

a) Derive the weak form for this type of problems and give explicit expressions for the element stiffness matrix and element load vector. Assume parts of the boundary with essential boundary conditions and the rest with the above natural boundary condition. How this last condition affects the stiffness matrix and load vector?

b) Now consider a series of heating cables placed in a conducting medium, as shown in the figure. The medium has conductivity of $k_x = 10$ W/(cm·°C) and $k_y = 15$ W/(cm·°C), the upper surface is exposed to a temperature of -5 °C, and the lower surface is bounded by an insulating medium. Assume that each cable is a point source of 250 W/cm. Take the convection coefficient between the medium and the upper surface to be $\beta = 5$ W/(cm^2·°C). Use a 8×8 mesh of linear quadrilateral elements in the computational domain (use any symmetry available in the problem). Plot the temperature contour. Check the convergence of the mesh.



Hint: Using symmetry of the problem, we can reduce the computational domain to that shown in the figure. The heat input at the node where the cable is located is 125 W/cm located at node 63. The boundary conditions at the upper boundary is that of convective type, at the right and left boundaries the heat flux is zero (because of symmetry), and at the lower boundary the heat flux is zero because of the insulation.