**Due Monday, September 28th, 12:00 midnight**

All homework problems here refer to the solution of one-dimensional boundary value problems: Weak forms, issues of numerical integration, discretization, boundary conditions, etc.

**Problem 1 – Properties of shape functions, finite element interpolation and interpolation error** (MatLab)

1. Modify the function FiniteElement_1D.m to include quadratic and cubic shape functions in one-dimension.

2. Write a short MatLab program that uses the above function and for several values of $\xi$, $-1 \le \xi \le 1$ returns the following two values (for $N^e$=2,3,4, $N^e$=number of shape functions per element e)

$$a = \sum_{i=1}^{N^e} N_i^e(\xi),\; b = \sum_{i=1}^{N^e} \frac{dN_i^e(\xi)}{d\xi}$$

   Verify that a=1 and b=0. These tests are necessary (but not sufficient) to ensure correct calculation of the shape functions in FiniteElement_1D.m

3. Write a short MatLab program utilizing the function FiniteElement_1D.m to calculate values of the finite element interpolant of the function $g(x) = \sin \pi x,\, 0 \le x \le 1$. Use two elements with linear shape functions. Evaluate $g_h(x)$ and $g(x)$ at 10 equally spaced points in each element. Print the values of $x, g_h(x), g(x)$ and $|g_h(x) - g(x)|$ at each point. Note that since the shape functions $N_i^e(\xi)$ are written as functions of $\xi$, where $-1 \le \xi \le 1$, the program must select the value of $\xi$ at which calculations are to be made and then calculate the corresponding value of $x$ from $x = x_1^e + \frac{1}{2}(1+\xi)(x_2^e - x_1^e)$. Repeat the calculation for quadratic and cubic elements.

4. Using function ErrorAnalysis.m, extend the program in part 3 above to calculate the mean-square norm of the interpolation error. Calculate and print the value

$$\| g - g_h \| = \left( \int_0^1 \left[ g(x) - g_h(x) \right]^2 dx \right)^{1/2}$$

   for linear, quadratic and cubic elements. Use Gaussian integration of order 4.

**Problem 2 – Transverse displacement of a non-uniform cable** (Hand calculations and MatLab)

Consider the problem of finding the function u(x) that satisfies the differential equation

$$-\frac{d}{dx}(a\frac{du}{dx}) - f = 0, \quad 0 < x < 1$$

and the boundary conditions

$$u(0) = 0,\, u(1) = 0$$

where $a(x) = 1 + x,\ f(x) = 1 + 4x$.

This problem corresponds to the transverse displacement of a nonuniform cable fixed at both ends and subjected to a distributed transverse force. This problem is simple enough that an analytical solution exists: $u_{ex}(x) = x(1-x)$.

a) Determine the finite element solution to this problem using four linear finite elements over the domain $0 < x < 1$. Perform all calculations of the stiffness and force vector analytically, apply the boundary conditions and solve the resulting 3x3 system of equations.
b) Repeat the calculations by running the provided MATLAB code after you appropriately modify the *InputData.m*, *ff.m*, *qq.m*, and *pp.m* functions.
c) Repeat the calculations above for 4, 8, and 10 elements and compare the finite element solution and the derivative of the solution with the exact solution. Note that the analytical solution is quadratic in $x$ but these calculations use linear finite elements. Provide a clear summary of your results.

**Problem 3 – Axisymmetric radial heat flow through a circular cylinder** (MatLab)

Consider the axisymmetric radial heat flow by conduction through a homogeneous hollow cylinder of inside radius $r_i$ and outside radius $r_0$. If the cylinder is sufficiently long, the end effects can be neglected. The governing equation for the temperature is given as

$$-\frac{d}{dr}(kA\frac{dT}{dr}) = AQ, \quad r_i < r < r_0$$

where k is the thermal conductivity, $A = 2\pi rl$ the surface area, $l$ is the length of the cylinder, $Q$ the heat generation per unit area, $r$ the radial coordinate and T the temperature. The heat flux is given as

$$q = -kA\frac{dT}{dr}$$

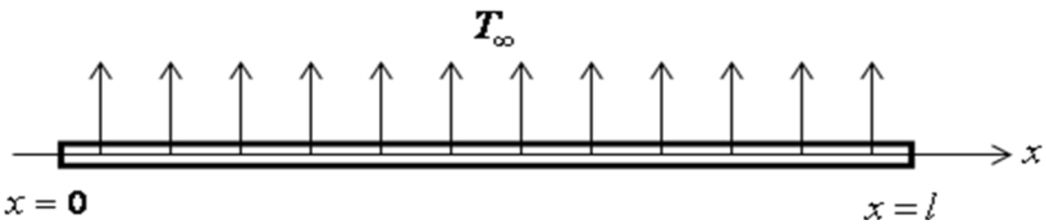
Use the data $r_0$ =3.25 in, $r_i$ =1.75 in, $l$ =10 ft, $Q$ = 0, $k$ = 0.04Btu/(h.ft.$^0$F), $T_i$ = 400$^o$F, $T_0$ = 80$^0$F, where $T_i$ and $T_0$ are the specified temperatures at the inner and outer walls, respectively.

Modify the given MATLAB programs to compute the temperature and flux in the cylinder (use 10 linear elements)

Compare your results with the analytical solution

$$T(r) = T_i - (T_i - T_0)\frac{\ln(r/r_i)}{\ln(r_0/r_i)}$$

**Problem 4 – Heat conduction with surface convection** (MatLab)




Develop the finite element equation $\mathbf{K}^e\mathbf{d}^e = \mathbf{f}^e$ for heat conduction with surface convection. The strong form in this case is given by

$$kA\frac{d^2T}{dx^2} = \beta h(T - T_\infty), \qquad 0 \le x \le l,$$

where $k, A = \pi r^2, h, \beta$ and $T_\infty$ are constants. $\beta = 2\pi r$ is the perimeter of the fin. To simplify the derivation, consider zero essential boundary conditions on both ends.

Now, let's consider also convection boundary equations

$$\frac{\partial T}{\partial x} = h(T - T_\infty) \quad \text{on} \quad x = l$$

Modifying the MatLab finite element code, solve the problem with the following parameters:

$k = 400$ W m$^{-1}$ $^\circ$C$^{-1}$, $l = 0.1$m, $h = 3000$ W m$^{-2}$ $^\circ$C$^{-1}$, $r = 10^{-2}$ m (radius of pin), $T_\infty = 20^\circ C$.

Boundary conditions: $T(0) = 80^\circ C$; $\frac{\partial T}{\partial x} = h(T - T_\infty)$ on $x = l$ .

Find the temperature and flux with uniform finite element meshes consisting of four, eight and sixteen 2-node linear elements.

**Problem 5 – Advection-diffusion problem** (MatLab)

Consider the one-dimensional advection-diffusion equation

$$v\frac{d\theta}{dx} - k\frac{d^2\theta}{dx^2} = 0, \quad 0 \le x \le 10$$

with the boundary conditions $\theta(0) = 0, \quad \theta(10) = 1$. Use linear finite elements and a 20 element mesh with uniformly space nodes such that the element length is $l^e = 0.5$. The Peclet number is defined as $P_e \equiv \frac{vl^e}{2k}$.

a. Solve the problem analytically in terms of $v$ and $k$.
b. Derive the weak form of this problem
c. Let $v = 2$ and $k = 5$ so that the Peclet number $P_e = 0.1$. Solve the problem by modifying the provided MatLab code. Compare the finite element solution with the exact solution.
d. Let $v = 60$ and $k = 5$ so that the Peclet number $P_e = 3.0$. Repeat the calculation. Compare your solution with the exact solution. Note that the solution will exhibit a spatial instability. For high values of the Peclet number, i.e. when advection dominates, special techniques must be developed to obtain accurate solution. We will introduce such methods later in this course.