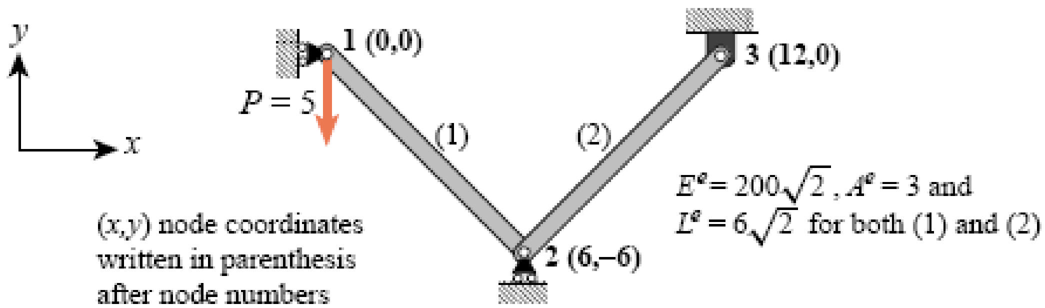
Tuesday October 27th, 7:30-9:30 pm

Problem 1 (20 points)



The plane truss problem defined in the Figure above has two elements and three nodes. Node 3 is fixed whereas 1 and 2 move over rollers as shown. The only nonzero applied load acts downward on node 1. Solve this problem by the Direct Stiffness Method. Start from the element stiffness equations given below. These are listed so you do not need to refer to the notes, and already incorporate the $E^e A^e/L^e$ factor in the stiffness matrices.

The element stiffness equations *in global coordinates* are

$$\begin{bmatrix} 50 & -50 & -50 & 50 \\ -50 & 50 & 50 & -50 \\ -50 & 50 & 50 & -50 \\ 50 & -50 & -50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} 50 & 50 & -50 & -50 \\ 50 & 50 & -50 & -50 \\ -50 & -50 & 50 & 50 \\ -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix}$$

(a) Assemble the master stiffness equations. (This result is reused in Question 3 below).
(b) Apply the given force and displacement BCs to get a reduced system of 2 equations and show it.
(c) Solve the reduced stiffness system for the unknown displacements and show the complete node displacement vector. *Skip recovery of node forces and reactions.*
(d) Recover the axial force $F^{(2)}$ in element (2) using the displacements you got in (c), noting sign.

Solution:

(a) Assembled master stiffness equations, obtained with any method:

$$\begin{bmatrix} 50 & -50 & -50 & 50 & 0 & 0 \\ -50 & 50 & 50 & -50 & 0 & 0 \\ -50 & 50 & 100 & 0 & -50 & -50 \\ -50 & 50 & 0 & 100 & -50 & -50 \\ 0 & 0 & -50 & -50 & 50 & 50 \\ 0 & 0 & -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}$$

(b) BCs: $u_{x1} = u_{y2} = u_{x3} = u_{y3} = 0$, $f_{y1} = -5$, $f_{x2} = 0$. Crossing out rows and columns 1, 4, 5 and 6 gives the reduced stiffness equation:

$$\begin{bmatrix} 50 & 50 \\ 50 & 100 \end{bmatrix} \begin{bmatrix} u_{y1} \\ u_{x2} \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

(c) Solving:
$u_{x2} = 1/10 = 0.10$, $u_{y1} = -1/5 = -0.20$
Complete displacement solution:

$$\mathbf{u} = [0 \quad -0.20 \quad 0.10 \quad 0 \quad 0 \quad 0]^T$$

Compute the internal bar force $F^{(2)}$ in element (2), which goes from node 2 to node 3. The orientation angle from x to 2→3 (+CCW) is 45◦. We have $c = \cos 45^\circ = 1/\sqrt{2}$ and $s = \sin 45^\circ = 1/\sqrt{2}$. The local displacements are recovered from the displacement transformation

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_{x2} = 1/10 \\ u_{y2} = 0 \\ u_{x3} = 0 \\ u_{y3} = 0 \end{bmatrix} = \begin{bmatrix} \bar{u}_{x2} = 1/(10\sqrt{2}) \\ * \\ \bar{u}_{x3} = 0 \\ * \end{bmatrix}$$

where * are values of no interest for this computation. The member elongation is

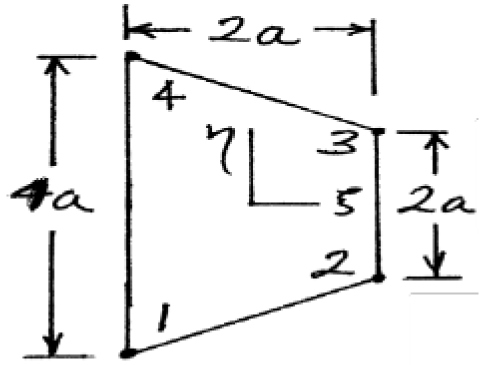
$$d^{(2)} = \bar{u}_{x3} - \bar{u}_{x2} = 0 - \left(\frac{1}{10\sqrt{2}}\right) = -\frac{1}{10\sqrt{2}}$$

So finally:

$$F^{(2)} = \frac{E^{(2)}A^{(2)}}{L^{(2)}} d^{(2)} = \frac{600\sqrt{2}}{6\sqrt{2}} \times -\frac{1}{10\sqrt{2}} = -5\sqrt{2} = -7.07 \text{ (C)}$$

Problem 2 (10 points)

Derive the Jacobian matrix (in terms of ξ and η) for the following 4-noded quadrilateral element



Note:

The basis functions in natural coordinates are:

$$N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta), N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta), N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

Solution:

$$\left[J\right]=\frac{1}{4}\begin{bmatrix}-(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta)\\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi)\end{bmatrix}\begin{bmatrix}-a & -2a\\ a & -a\\ a & a\\ -a & 2a\end{bmatrix}=\begin{bmatrix}a & -\frac{a}{2}\eta\\ 0 & \frac{a}{2}(3-\xi)\end{bmatrix}$$

Note that

$$\det\left[J\right]=\frac{a^2}{2}(3-\xi)$$

The det[J] is a function of only ξ but [J] depends on both ξ and η.

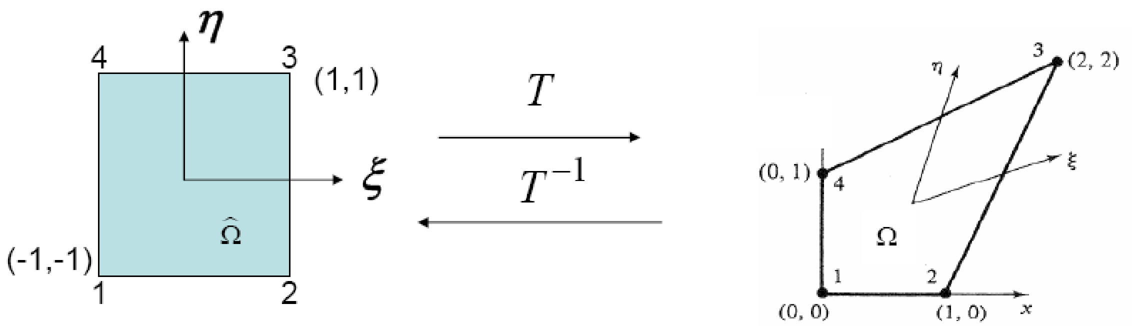
Problem 3 (10 points)

What is the convergence rate of the bending stresses in a beam element using the Hermite polynomials as the interpolation functions and why?

Solution

The displacement field in a beam is cubic, so that displacement converge with the fourth power of the mesh. Stresses are proportional to the second derivatives (curvatures), and so they converge like the square of the mesh.

Problem 4 (15 points) – For the 4-node element Ω shown in the figure below, compute $\frac{\partial N_2}{\partial x}$ at the point $(\xi,\eta)=(0.5,0.5)$ in Ω.



Solution

The mapping here is given as:

$$x = N_2 + 2N_3 = \frac{1}{4}(3+3\xi+\eta+\xi\eta)$$

$$y = N_4 + 2N_3 = \frac{1}{4}(3+\xi+3\eta+\xi\eta)$$

Therefore, the Jacobian matrix is

$$J^T = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3+\eta & 1+\xi \\ 1+\eta & 3+\xi \end{bmatrix}_{\xi=0.5,\eta=0.5} = \begin{bmatrix} \frac{7}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{7}{8} \end{bmatrix}$$

The determinant is $|J| = \frac{5}{8}$.

From the formulae derived in class,

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta}, \quad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta},$$
$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}$$

we have

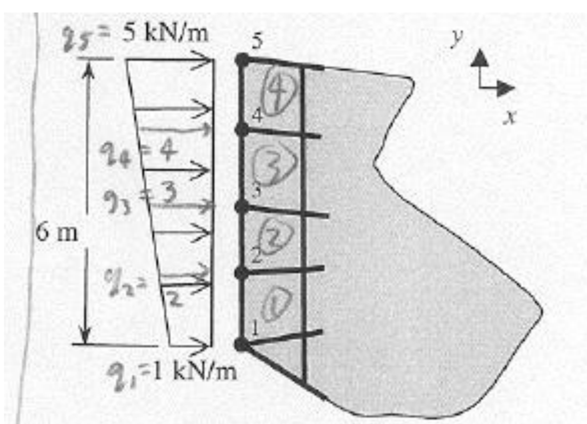
$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta} = \frac{8}{5}\frac{7}{8} = \frac{7}{5}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi} = -\frac{8}{5}\left(\frac{3}{8}\right) = -\frac{3}{5}$$
$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta)\Big|_{\eta=0.5} = \frac{1}{8}, \quad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi)\Big|_{\xi=0.5} = -\frac{3}{8}$$

Finally, we can get

$$\frac{\partial N_2}{\partial x} = \frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_2}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{8}\times\left(\frac{7}{5}\right) + \left(-\frac{3}{8}\right)\times\left(-\frac{3}{5}\right) = \frac{2}{5}$$

Problem 5 (15 points)

Find the finite element equivalent nodal load $F = \left(F_{1x} \quad F_{1y} \quad .. \quad .. \quad F_{5x} \quad F_{5y}\right)^T$ for the distributed load shown in the Figure. The load is applied in the x-direction. Four node quadrilateral elements with boundary sides of equal length are used.



Solution

$$\begin{pmatrix} F_{1x}^{(e)} \\ F_{2x}^{(e)} \end{pmatrix} = \int_{-1}^{1} (q_1^e \frac{1-\xi}{2} + q_2^e \frac{1+\xi}{2}) \begin{pmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{pmatrix} \frac{L^e}{2} d\xi = \frac{L^e}{2} \begin{pmatrix} \int_{-1}^{1} \left(\frac{1-\xi}{2}\right)^2 d\xi & \int_{-1}^{1} \frac{1-\xi^2}{4} d\xi \\ \int_{-1}^{1} \frac{1-\xi^2}{4} d\xi & \int_{-1}^{1} \left(\frac{1+\xi}{2}\right)^2 d\xi \end{pmatrix} \begin{pmatrix} q_1^e \\ q_2^e \end{pmatrix} = \frac{L^e}{2} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} q_1^e \\ q_2^e \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} F_{1x}^{(e)} \\ F_{2x}^{(e)} \end{pmatrix} = \frac{L^e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_1^e \\ q_2^e \end{pmatrix}$$

$$\begin{pmatrix} F_{1x}^{(1)} \\ F_{2x}^{(1)} \end{pmatrix} = \begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix} = \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{6/4}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.25 \end{pmatrix} kN$$

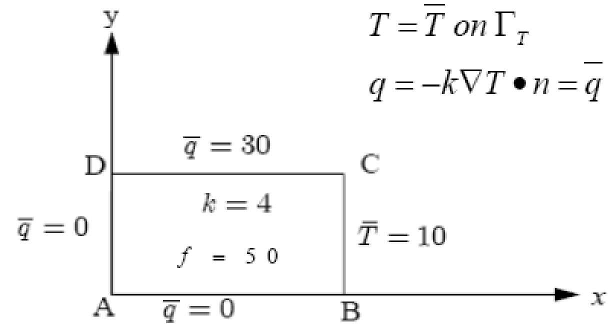
$$\begin{pmatrix} F_{1x}^{(2)} \\ F_{2x}^{(2)} \end{pmatrix} = \begin{pmatrix} F_{2x} \\ F_{3x} \end{pmatrix} = \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1.75 \\ 2 \end{pmatrix} kN$$

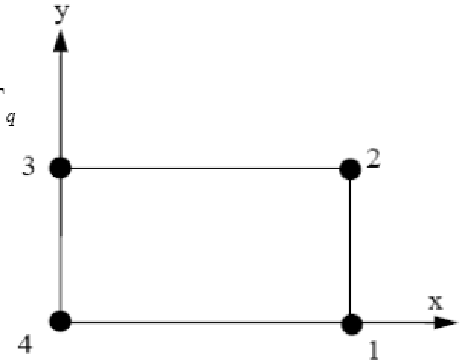
$$\begin{pmatrix} F_{1x}^{(3)} \\ F_{2x}^{(3)} \end{pmatrix} = \begin{pmatrix} F_{3x} \\ F_{4x} \end{pmatrix} = \frac{6/4}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.75 \end{pmatrix} kN \quad \begin{pmatrix} F_{1x}^{(4)} \\ F_{2x}^{(4)} \end{pmatrix} = \begin{pmatrix} F_{4x} \\ F_{5x} \end{pmatrix} = \frac{6/4}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3.25 \\ 3.5 \end{pmatrix} kN$$

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \\ F_{5x} \\ F_{5y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1.25+1.75 \\ 0 \\ 2+2.5 \\ 0 \\ 2.75+3.25 \\ 0 \\ 3.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 4.5 \\ 0 \\ 6 \\ 0 \\ 3.5 \\ 0 \end{pmatrix} kN$$

Problem 6 (30 points) - Consider a heat conduction problem on a rectangular (2 x 1) m domain as shown in the Figure below.

$$-\nabla \bullet (k\nabla T) = f(x,y)$$
$$T = \overline{T} \; on \; \Gamma_T$$
$$q = -k\nabla T \bullet n = \overline{q} \; on \; \Gamma_q$$





The conductivity is $k=4$ W^{o}C^{-1}, $\overline{T}$ = 10 ^{o}C is prescribed along edge BC. Edges AB and AD are insulated, i.e. $\overline{q}$ = 0 W m^{-1} ; along edge DC the boundary flux is $\overline{q}$ = 30 W m^{-1} . A constant heat source is given f = 50 W m^{-2} . We will analyze this problem using **one** 4-node finite element. Use as (local and global) node numbering the one shown on the right of the Figure.

a) Provide a precise weak statement for the problem and based on this give the final expressions for the element stiffness matrix and load vectors.

b) Compute the element stiffness matrix. This will require a number of tasks summarized for your convenience below

   1. Using the basis functions, give an expression in terms of (ξ,η) for the Jacobian matrix
   $$J^e = \begin{bmatrix} \dfrac{\partial x^e}{\partial \xi} & \dfrac{\partial y^e}{\partial \xi} \\ \dfrac{\partial x^e}{\partial \eta} & \dfrac{\partial y^e}{\partial \eta} \end{bmatrix}$$

   2. Compute expressions for the determinant of the Jacobian and the inverse of this Jacobian matrix.

   3. Compute the matrix of the derivates of the shape functions wrt natural coordinates, i.e.

$$\begin{bmatrix} \dfrac{\partial N_1^e}{\partial \xi} & \dfrac{\partial N_2^e}{\partial \xi} & \dfrac{\partial N_3^e}{\partial \xi} & \dfrac{\partial N_4^e}{\partial \xi} \\ \dfrac{\partial N_1^e}{\partial \eta} & \dfrac{\partial N_2^e}{\partial \eta} & \dfrac{\partial N_3^e}{\partial \eta} & \dfrac{\partial N_4^e}{\partial \eta} \end{bmatrix}$$

4. Give an expression for the B^e matrix relating $\begin{bmatrix} \dfrac{\partial T^e}{\partial x} \\ \dfrac{\partial T^e}{\partial y} \end{bmatrix}$ and the nodal temperatures.

5. Show in as much detail as possible (give the final expression but you don't need to do the actual detailed calculation) how you will use Gauss quadrature to compute the stiffness matrix and load vector. Since you only have one element, this will also be your global stiffness and load vector.

Note: The location of the 4 Gauss points for 2D integration is $(\xi,\eta) = (\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$ and each weight is equal to 1.

For the purposes of question (d) below use the following stiffness matrix.

$$\begin{bmatrix} 3.3333 & -2.3333 & -1.6667 & 0.6667 \\ -2.3333 & 3.3333 & 0.6667 & -1.6667 \\ -1.6667 & 0.6667 & 3.3333 & -2.3333 \\ 0.6667 & -1.6667 & -2.3333 & 3.3333 \end{bmatrix}$$

c) Compute the element load vector f^e. There are three contributions – one from the heat source f_Ω^e, another f_q^e from the natural boundary condition and the third (unknown) from the normal heat flux at the nodes with prescribed temperature (`reaction fluxes').

1. Compute the first term f_Ω^e <u>either</u> analytically or using 2D Gauss integration.

2. Compute the load contribution f_q^e from the flux boundary condition. This requires boundary (one-dimensional) integration that for this problem you can perform analytically (or using common sense!)

d) Using the computed load vector and the provided stiffness matrix, apply essential BCs and **solve for the unknown nodal temperatures**.

e) Compute the heat flux components at the Gauss point $(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})$.

Solution:

(a) The weak statement for this problem takes the form:
Find a function $T(x,y)\in H^1(\Omega)$, with $T(s)=\overline{T}(s), s\in\Gamma_T$ such that for any $w\in H^1(\Omega)$ with $w=0$ on Γ_T the following equations holds:

$$\int_\Omega k\nabla T\cdot\nabla w d\Omega=\int_\Omega fwd\Omega-\int_{\Gamma_q}\overline{q}wd\Gamma$$

From the weak form, we can easily identify:

$$K^e=\int_{\Omega^e}kB^TBd\Omega,\quad f^e=\int_{\Omega^e}N^T f\,d\Omega-\int_{\Gamma_q^e}N^T\overline{q}\,d\Gamma$$

(b) 1. The basis functions in natural coordinates are:

$$N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta), N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta), N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

Then the mapping is

$$x=\sum_{i=1}^{4}x_iN_i=2N_1+2N_2=1-\eta$$

$$y=\sum_{i=1}^{4}y_iN_i=N_2+N_3=\frac{1}{2}(1+\xi)$$

The Jacobian matrix can be obtained as

$$J=\begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{2}\\ -1 & 0\end{bmatrix}$$

2. The determinant of the Jacobian is $|J|=\frac{1}{2}$.

The inverse of this Jacobian matrix is

$$J^{-1}=2\begin{bmatrix}0 & -\frac{1}{2}\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$$

3. $$\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -1-\xi & 1+\xi & 1-\xi \end{bmatrix}$$

4.

$$\left[B^e\right] = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial \xi} & \frac{\partial \widehat{N}_2^e}{\partial \xi} & \frac{\partial \widehat{N}_3^e}{\partial \xi} & \frac{\partial \widehat{N}_4^e}{\partial \xi} \\ \frac{\partial \widehat{N}_1^e}{\partial \eta} & \frac{\partial \widehat{N}_2^e}{\partial \eta} & \frac{\partial \widehat{N}_3^e}{\partial \eta} & \frac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix} = \frac{1}{|J^e|} \begin{bmatrix} \frac{\partial y^e}{\partial \eta} & -\frac{\partial y^e}{\partial \xi} \\ -\frac{\partial x^e}{\partial \eta} & \frac{\partial x^e}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial \xi} & \frac{\partial \widehat{N}_2^e}{\partial \xi} & \frac{\partial \widehat{N}_3^e}{\partial \xi} & \frac{\partial \widehat{N}_4^e}{\partial \xi} \\ \frac{\partial \widehat{N}_1^e}{\partial \eta} & \frac{\partial \widehat{N}_2^e}{\partial \eta} & \frac{\partial \widehat{N}_3^e}{\partial \eta} & \frac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix}$$

This comes from an earlier derivation:

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \frac{1}{|J^e|} \begin{bmatrix} \frac{\partial y^e}{\partial \eta} & -\frac{\partial x^e}{\partial \eta} \\ -\frac{\partial y^e}{\partial \xi} & \frac{\partial x^e}{\partial \xi} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

$$B = J^{-1} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1-\xi & 1+\xi & -1-\xi & \xi-1 \\ 2(\eta-1) & 2(1-\eta) & 2(1+\eta) & -2(\eta+1) \end{bmatrix}$$

5. The stiffness matrix is

$$K = \int_\Omega kB^T B d\Omega = \int_{-1}^{1}\int_{-1}^{1} kB^T B\,|J|\,d\xi d\eta = \sum_{i=1}^{2}\sum_{i=1}^{2} k\left(B^T B\,|J|\right)_{\xi=\xi_i,\eta=\eta_i} W_i W_j$$

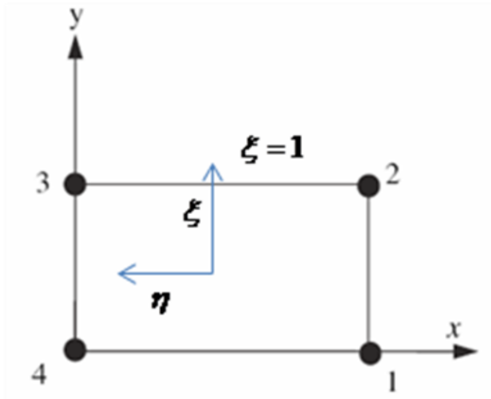
$$= \begin{bmatrix} 3.3333 & -2.3333 & -1.6667 & 0.6667 \\ -2.3333 & 3.3333 & 0.6667 & -1.6667 \\ -1.6667 & 0.6667 & 3.3333 & -2.3333 \\ 0.6667 & -1.6667 & -2.3333 & 3.3333 \end{bmatrix}$$

(c) 1.

$$f_\Omega^e = \int_{-1}^{1}\int_{-1}^{1} f\,N^T\,|J|\,d\xi d\eta = \begin{bmatrix} 25 \\ 25 \\ 25 \\ 25 \end{bmatrix}$$

2. We now need to compute the contribution to element load from the natural boundary conditions along edge CD. When using natural coordinate, this side is mapped to the side

2-3 on the master element (paying close attention to the given numbering of the nodes). Then we have (see Fig below) $\xi = 1$



$$f_q^e = -\int_{-1}^{1} \bar{q}\left(N^T \mid J \mid\right)_{\xi=1} d\eta = -30\int_{-1}^{1} \begin{bmatrix} 0 \\ \frac{1}{2}(1-\eta) \\ \frac{1}{2}(1+\eta) \\ 0 \end{bmatrix} \frac{2-0}{2} d\eta = \begin{bmatrix} 0 \\ -30 \\ -30 \\ 0 \end{bmatrix}$$

(d) Assemble all the node contributions give:

$$\begin{bmatrix} 3.3333 & -2.3333 & -1.6667 & 0.6667 \\ -2.3333 & 3.3333 & 0.6667 & -1.6667 \\ -1.6667 & 0.6667 & 3.3333 & -2.3333 \\ 0.6667 & -1.6667 & -2.3333 & 3.3333 \end{bmatrix} \begin{bmatrix} T_1 = 10 \\ T_2 = 10 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} r_1 + 25 \\ r_2 - 5 \\ -5 \\ 25 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 3.3333 & -2.3333 \\ -2.3333 & 3.3333 \end{bmatrix} \begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 25 \end{bmatrix} - \begin{bmatrix} -2.3333 & -1.6667 \\ -1.6667 & -2.3333 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 35 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 17.3529 \\ 22.6471 \end{bmatrix} {}^\circ C$$

(e) The heat flux components at the Gauss point $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are

$$q = \begin{bmatrix} -k\dfrac{\partial T}{\partial x} \\ -k\dfrac{\partial T}{\partial y} \end{bmatrix} = -kB_{\xi=\frac{1}{\sqrt{3}},\eta=-\frac{1}{\sqrt{3}}} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = -4\begin{bmatrix} 0.1057 & 0.3943 & -0.3943 & -0.1057 \\ -0.7887 & 0.7887 & 0.2113 & -0.2113 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 17.3529 \\ 22.6471 \end{bmatrix}$$

$$= \begin{bmatrix} 16.9434 \\ 4.4752 \end{bmatrix} W/m$$