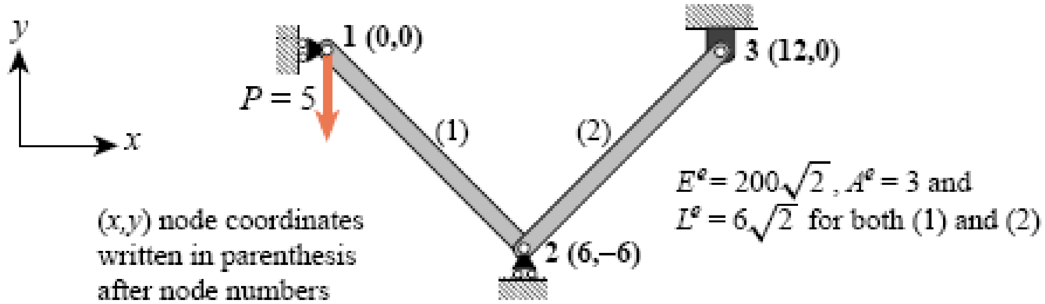
Tuesday October 27th, 7:30-9:30 pm

Problem 1 (20 points)



The plane truss problem defined in the Figure above has two elements and three nodes. Node 3 is fixed whereas 1 and 2 move over rollers as shown. The only nonzero applied load acts downward on node 1. Solve this problem by the Direct Stiffness Method. Start from the element stiffness equations given below (please note that they already incorporate the $E^e A^e/L^e$ factor in the stiffness matrices).

The element stiffness equations *in global coordinates* are

$$\begin{bmatrix} 50 & -50 & -50 & 50 \\ -50 & 50 & 50 & -50 \\ -50 & 50 & 50 & -50 \\ 50 & -50 & -50 & 50 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix} = \begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} 50 & 50 & -50 & -50 \\ 50 & 50 & -50 & -50 \\ -50 & -50 & 50 & 50 \\ -50 & -50 & 50 & 50 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix}$$

(a) Assemble the master stiffness equations. (This result is reused in Question 3 below).

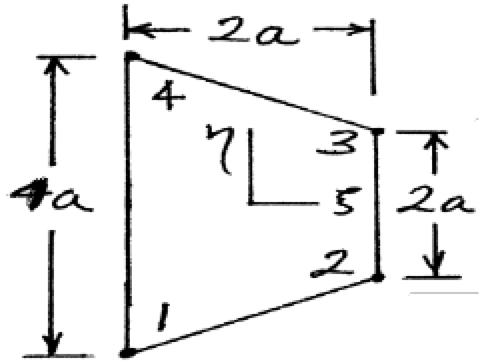
(b) Apply the given force and displacement BCs to get a reduced system of 2 equations and show it.

(c) Solve the reduced stiffness system for the unknown displacements and show the complete node displacement vector. *Skip recovery of node forces and reactions.*

(d) Recover the axial force $F^{(2)}$ in element (2) using the displacements you got in (c), noting sign.

Problem 2 (10 points)

Derive the Jacobian matrix (in terms of ξ and η) for the following 4-noded quadrilateral element



Note:

The basis functions in natural coordinates are:

$$\widehat{N}_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta),\widehat{N}_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

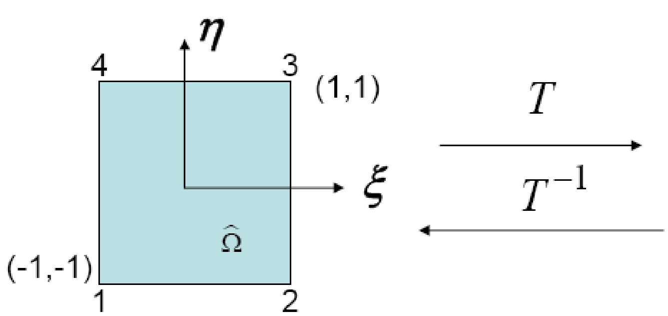
$$\widehat{N}_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta),\widehat{N}_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

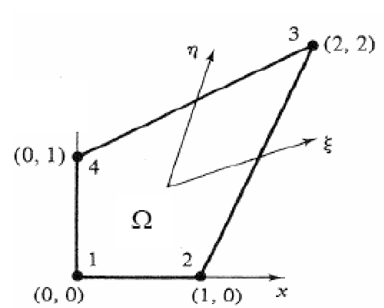
Problem 3 (10 points)

What is the convergence rate of the bending stresses in a beam element using the Hermite polynomials as the interpolating functions and why?

Problem 4 (15 points)

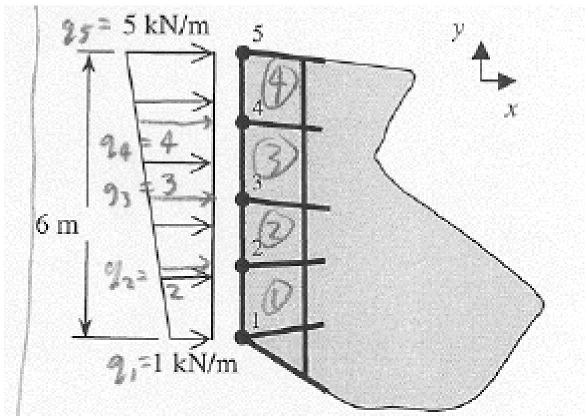
For the 4-node element Ω shown in the figure below, compute $\frac{\partial N_2}{\partial x}$ at the point $(\xi,\eta)=(0.5,0.5)$ in $\widehat{\Omega}$.





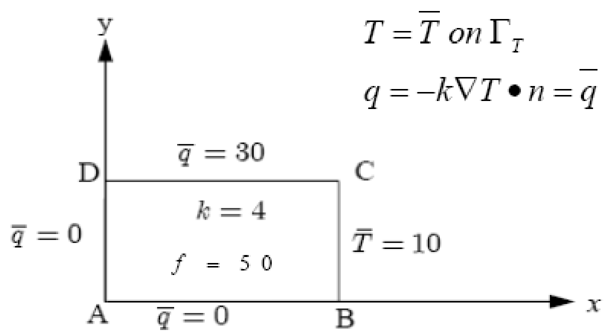
Problem 5 (15 points)

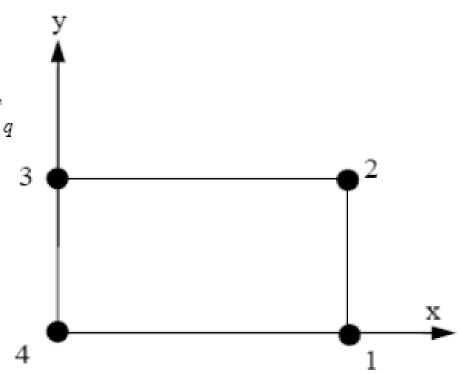
Find the finite element equivalent nodal load $F = \left(F_{1x} \quad F_{1y} \quad .. \quad .. \quad F_{5x} \quad F_{5y} \right)^T$ for the distributed load shown in the Figure. **The load is applied in the x-direction.** Four node quadrilateral elements with boundary sides of equal length are used.



Problem 6 (30 points) - Consider a heat conduction problem on a rectangular (2 x 1) m domain as shown in the Figure below.

$$-\nabla \bullet (k \nabla T) = f(x, y)$$
$$T = \overline{T} \; on \; \Gamma_T$$
$$q = -k \nabla T \bullet n = \overline{q} \; on \; \Gamma_q$$





The conductivity is $k=4$ W^{o}C^{-1}, $\overline{T}$ = 10 ^{o}C is prescribed along edge BC. Edges AB and AD are insulated, i.e. $\overline{q}$ = 0 W m^{-1} ; along edge DC the boundary flux is $\overline{q}$ = 30 W m^{-1} . A constant heat source is given f = 50 W m^{-2} . We will analyze this problem using **one** 4-node finite element. Use as (local and global) node numbering the one shown on the right of the Figure.

a) Provide a precise weak statement for the problem and based on this give the final expressions for the element stiffness matrix and load vectors.

b) Compute the element stiffness matrix. This will require a number of tasks summarized for your convenience below

1. Using the basis functions, give an expression in terms of (ξ,η) for the Jacobian matrix

$$J^e = \begin{bmatrix} \dfrac{\partial x^e}{\partial \xi} & \dfrac{\partial y^e}{\partial \xi} \\ \dfrac{\partial x^e}{\partial \eta} & \dfrac{\partial y^e}{\partial \eta} \end{bmatrix}$$

2. Compute expressions for the determinant of the Jacobian and the inverse of this Jacobian matrix.

3. Compute the matrix of the derivates of the shape functions wrt natural coordinates, i.e.

$$\begin{bmatrix} \dfrac{\partial \widehat{N}_1^e}{\partial \xi} & \dfrac{\partial \widehat{N}_2^e}{\partial \xi} & \dfrac{\partial \widehat{N}_3^e}{\partial \xi} & \dfrac{\partial \widehat{N}_4^e}{\partial \xi} \\ \dfrac{\partial \widehat{N}_1^e}{\partial \eta} & \dfrac{\partial \widehat{N}_2^e}{\partial \eta} & \dfrac{\partial \widehat{N}_3^e}{\partial \eta} & \dfrac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix}$$

4. Give an expression for the B^e matrix relating $\begin{bmatrix} \dfrac{\partial T^e}{\partial x} \\ \dfrac{\partial T^e}{\partial y} \end{bmatrix}$ and the element nodal temperatures.

5. Show in as much detail as possible (give the final expression but you don't need to do the actual detailed calculation) how you will use Gauss quadrature to compute the stiffness matrix and load vector. Since you only have one element, this will also be your global stiffness and load vector.

Note: The location of the 4 Gauss points for 2D integration is $(\xi,\eta) = (\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$ and each weight is equal to 1.

For the purposes of question (d) below use the following stiffness matrix.

$$\begin{bmatrix} 3.3333 & -2.3333 & -1.6667 & 0.6667 \\ -2.3333 & 3.3333 & 0.6667 & -1.6667 \\ -1.6667 & 0.6667 & 3.3333 & -2.3333 \\ 0.6667 & -1.6667 & -2.3333 & 3.3333 \end{bmatrix}$$

c) Compute the element load vector f^e. There are three contributions – one from the heat source f_Ω^e, another f_q^e from the natural boundary condition and the third (unknown) from the normal heat flux at the nodes with prescribed temperature (`reaction fluxes').

1. Compute the first term f_Ω^e <u>either</u> analytically or using 2D Gauss integration.
2. Compute the load contribution f_q^e from the flux boundary condition. This requires boundary (one-dimensional) integration that for this problem you can perform analytically (or using common sense!)

d) Using the computed load vector and the provided stiffness matrix, apply essential BCs and **solve for the unknown nodal temperatures**.

e) Compute the heat flux components at the Gauss point $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.