MAE 212: Spring 2001
Lecture 9
N. Zabaras

YIELD CRITERIA

Yield criteria

Recall that we say that the material yields when it exhibits an irreversible straining which is sustained once a certain level of the stress distribution is reached.

A yield criterion indicates for which combination of stress components transition from elastic (recoverable) to plastic (permanent) deformations occurs.

We will start our discussion with initial yielding and then proceed to discuss how material yielding is sustained. In one-dimension (Fig. 1(a)) yielding occurs when the uniaxial stress reaches the value of the yield stress Y in tension, i.e. at $\sigma = Y$. When does 'yielding' occurs in multi-axial stress states? (Fig. 1(b))? The answer is given with phenomenological theories called 'yield criteria'. Instead of presenting the requirements and constraints for a general form of a yield criterion, we will here only examine the two most important yield criteria for isotropic materials.

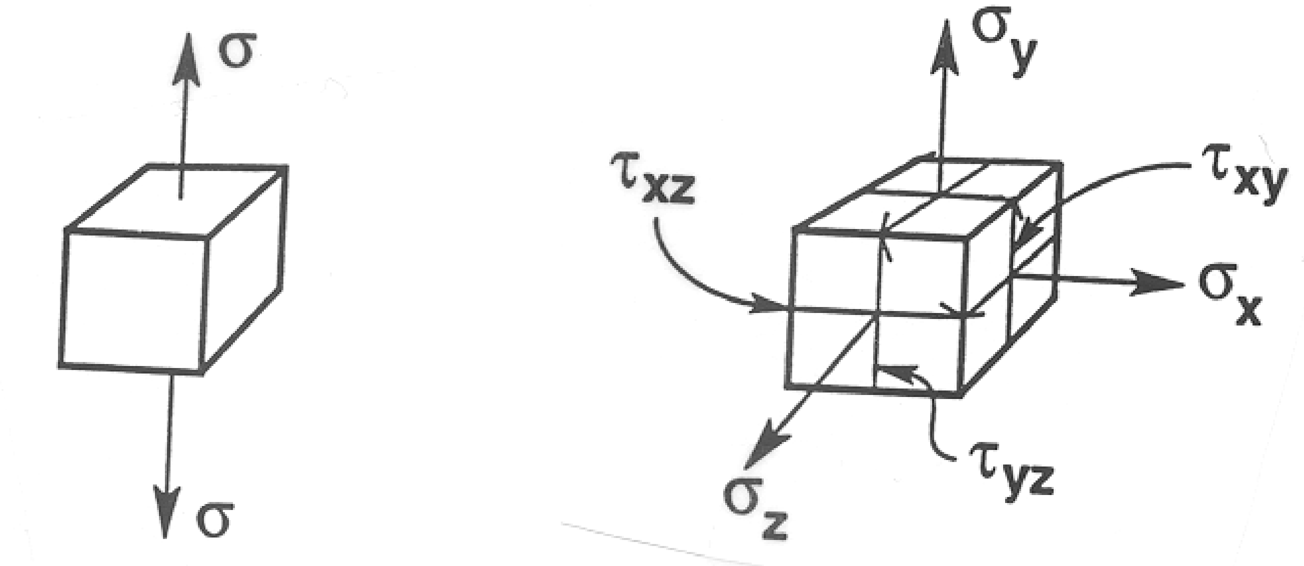

Figure 1: (a) To define yielding in one-dimensional stress states, we compare the uniaxial stress σ with the yield stress in tension Y. (b) When does yielding occurs in multi-dimensions?

The Tresca criterion or the Maximum shear stress criterion

According to Tresca, in the general multi-dimensional stress state, yielding occurs when:

$$\tau_{\max} = \kappa, \qquad \text{where } \kappa \text{ is the yield stress in shear} \tag{1}$$

Recall that:

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \tag{2}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximum and minimum principal stresses, respectively. The yield stress κ can be understood as the shear stress level τ in a pure shear test at which transition from recoverable to non-recoverable shear strains γ occurs. Think of the diagram τ versus γ obtained in a pure shear test the same way you think of the diagram σ versus ϵ in tension. The yield stress κ in shear can be defined the same way Y was defined in a tensile test.

The yield stress κ in shear is not independent of the yield stress Y in uniaxial tension. To compute their relation, apply the Tresca criterion to uniaxial tension. For this case, $\sigma_1 \neq 0$ and $\sigma_2 = \sigma_3 = 0$. So we can write:

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{\sigma_1 - 0}{2} = \frac{\sigma_1}{2} = \kappa \tag{3}$$

from which we conclude that in uniaxial tension yielding occurs when $\sigma_1 = 2\kappa$, i.e.

$$\text{For the Tresca criterion :} \qquad \kappa = \frac{Y}{2} \tag{4}$$

Using the above expression, we can summarize the Tresca yield criterion as follows:

$$\text{In a general stress state yielding occurs when :} \qquad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \kappa = \frac{Y}{2} \tag{5}$$

Figure 2 shows a graphical representation of this criterion for the case of plane stress ($\sigma_2 = 0$). To verify the form of this diagram the only thing you need to account carefully is the values of $\sigma_{\max}$ and $\sigma_{\min}$ in each region of the $\sigma_1 - \sigma_3$ plane.

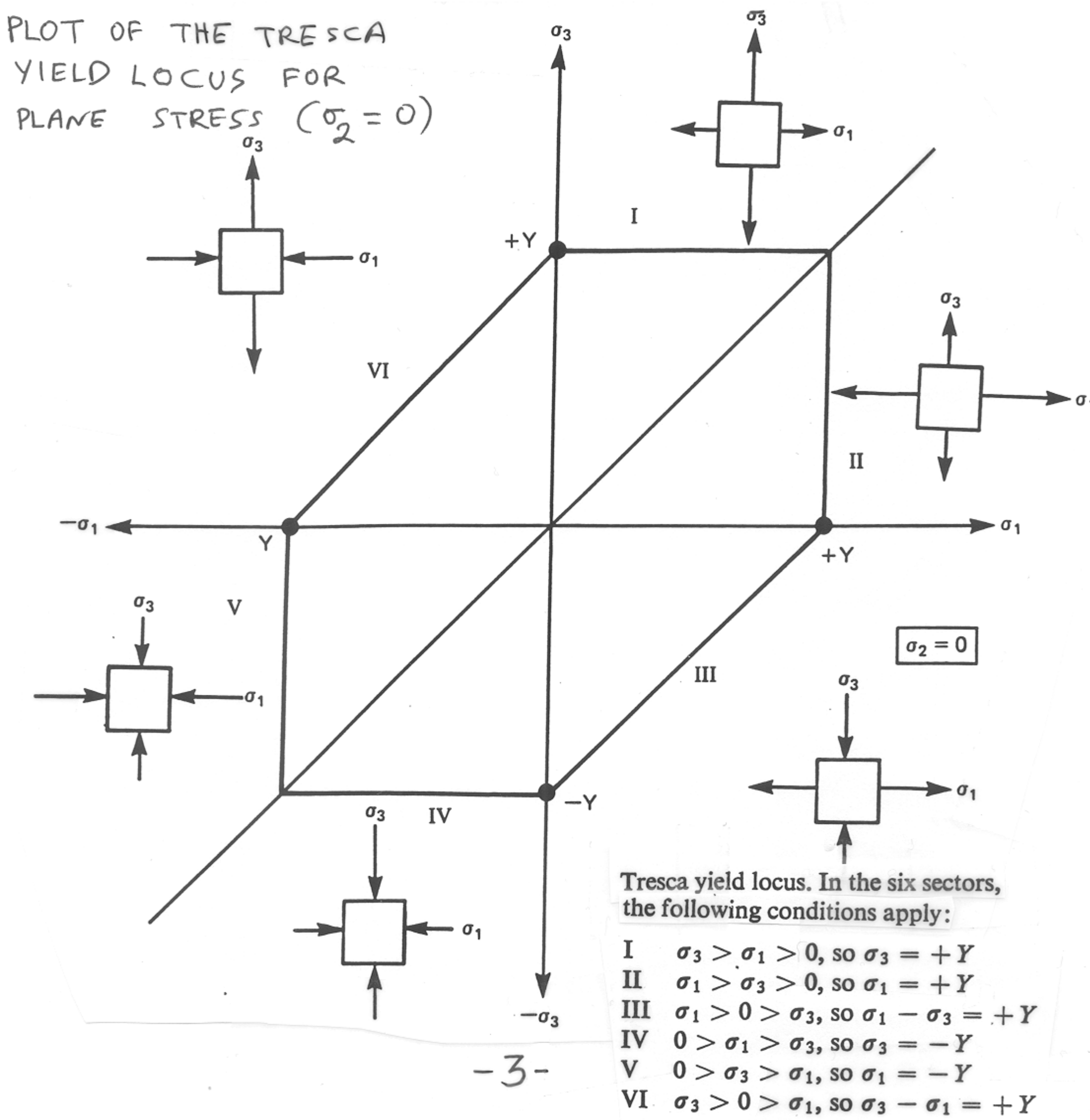

Figure 2: Plot of the Tresca yield locus for the case of plane stress ($\sigma_2 = 0$). In each of the six sectors of this diagram you need to find the σ_{max} and σ_{min} from the principal stress components σ_1, $\sigma_2 = 0$ (plane stress) and σ_3.

Examples of the application of the Tresca criterion

Example 1: Equi-biaxial tension $\sigma_1 = \sigma_2 = \sigma$ and $\sigma_3 = 0$. (Fig. 3)

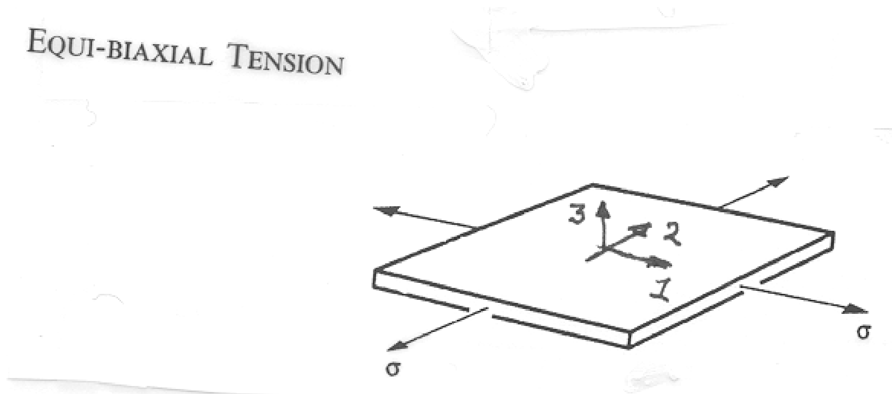

Figure 3: Equi-biaxial tension: $\sigma_1 = \sigma_2 = \sigma$ and $\sigma_3 = 0$.

The principal stresses are:

$$\sigma_{\max} = \sigma_1 = \sigma \tag{6}$$
$$\sigma_{\min} = \sigma_3 = 0 \tag{7}$$

$$\tau_{\max} = \frac{\sigma - 0}{2} = \frac{Y}{2} \tag{8}$$

from which we conclude that yielding in equi-biaxial tension occurs when $\sigma = Y$.

Example 2: Hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3 = -p$ (Fig. 4)

The principal stresses are:

$$\sigma_{\max} = -p \tag{9}$$
$$\sigma_{\min} = -p \tag{10}$$

$$\tau_{\max} = \frac{(-p) - (-p)}{2} = 0 \neq \frac{Y}{2} \tag{11}$$

So since $\tau_{\max} \neq \frac{Y}{2}$ regardless of the value of p, we conclude that yielding can never occur in a purely hydrostatic stress state (Important note: In reality this and other yield criteria are designed to start with such that a hydrostatic stress state leads to non-yielding as it is observed in experiments!).

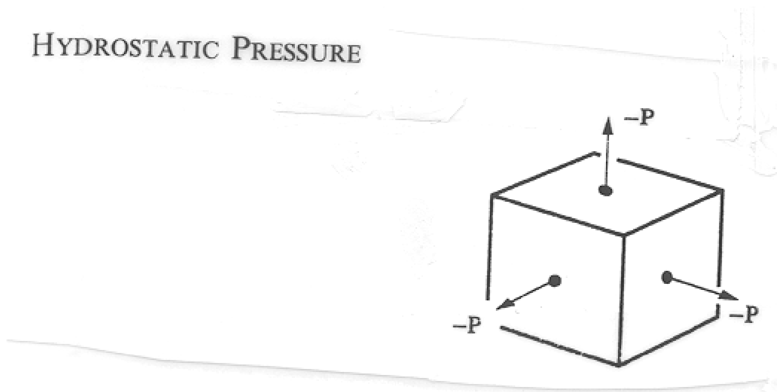

Figure 4: Hydrostatic pressure, $\sigma_1 = \sigma_2 = \sigma_3 = -p$.

Example 3: Thin pressurized tube with end caps ($\frac{r}{t} > 10$) (Fig. 5)

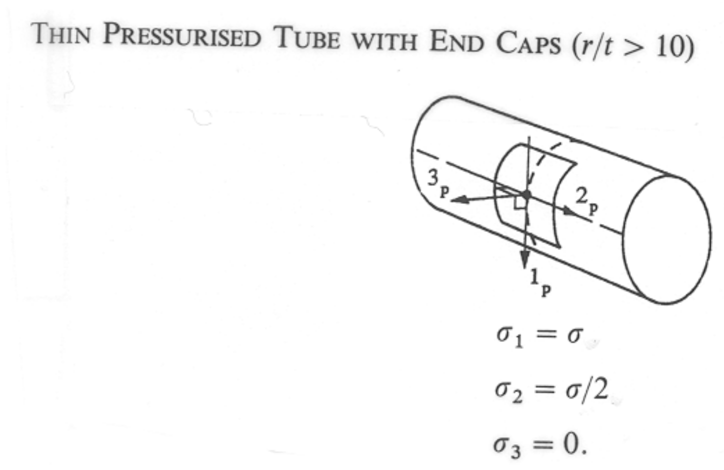

Figure 5: Thin wall pressurized tube under pressure p with the principal directions denoted as $1_p = \theta$, $2_p = z$ and $3_p = r$ in terms of the θ, z, r coordinate system seen earlier in the course.

We have seen in several occasions earlier in this course that the principal stresses are:

$$\sigma_{\mathrm{r}} = 0 \tag{12}$$

$$\sigma_\theta = \frac{pr}{t} \tag{13}$$

$$\sigma_z = \frac{pr}{2t} \tag{14}$$

Using these equations we conclude that:

$$\sigma_{\mathrm{max}} = \sigma_\theta = \frac{pr}{t} \tag{15}$$

$$\sigma_{\min} = \sigma_{\mathrm{r}} = 0 \tag{16}$$

So substitution to the Tresca criterion gives:

$$\tau_{\max} = \frac{\frac{pr}{t} - 0}{2} = \frac{Y}{2} \tag{17}$$

from which we conclude that yielding occurs for $\frac{pr}{t} = Y$ or for $p = \frac{tY}{r}$.

Example 4: Pure shear: $\sigma_1 = -\sigma_2 = \tau$ and $\sigma_3 = 0$ (Fig. 6)

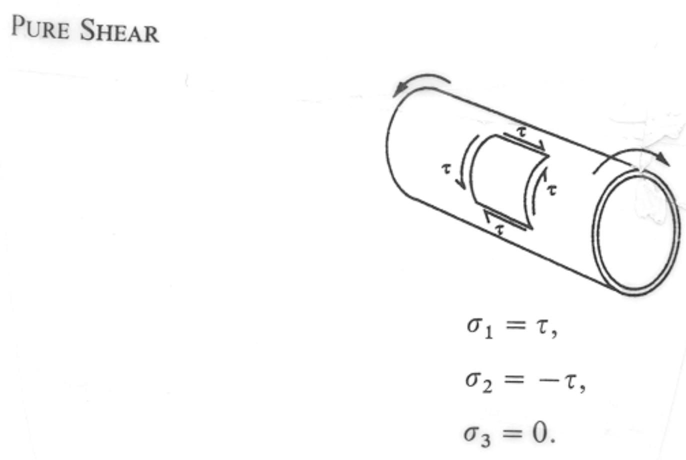

Figure 6: Pure shear state: $\sigma_1 = -\sigma_2 = \tau$, $\sigma_3 = 0$.

Substitution to the Tresca criterion gives:

$$\tau_{\max} = \frac{\tau - (-\tau)}{2} = \tau, \quad \text{so yielding occurs when :} \quad \tau = \frac{Y}{2} \tag{18}$$

which we already have seen earlier in the form of $\kappa = \frac{Y}{2}$.

Example 5: General 3D case

Let us consider that a metal with a yield stress of 280 MPa is subjected to a stress state with principal stresses of 300 MPa, 200 MPa and 50 MPa. Will the metal yield based on the Tresca yield criterion?

Using the given stress values, we conclude that:

$$\sigma_{\max} = 300 \text{ MPa} \tag{19}$$
$$\sigma_{\min} = 50 \text{ MPa} \tag{20}$$

So substitution to the Tresca criterion gives:

$$\tau_{\max} = \frac{300-50}{2} = \frac{250}{2} \neq \frac{280}{2} = \frac{Y}{2} \tag{21}$$

i.e. the metal will not yield.

Example 6: Yielding under general plane stress conditions (Fig. 7)

For the plane stress state of Figure 7, let us find the yield stress Y if we know that the material is yielding.

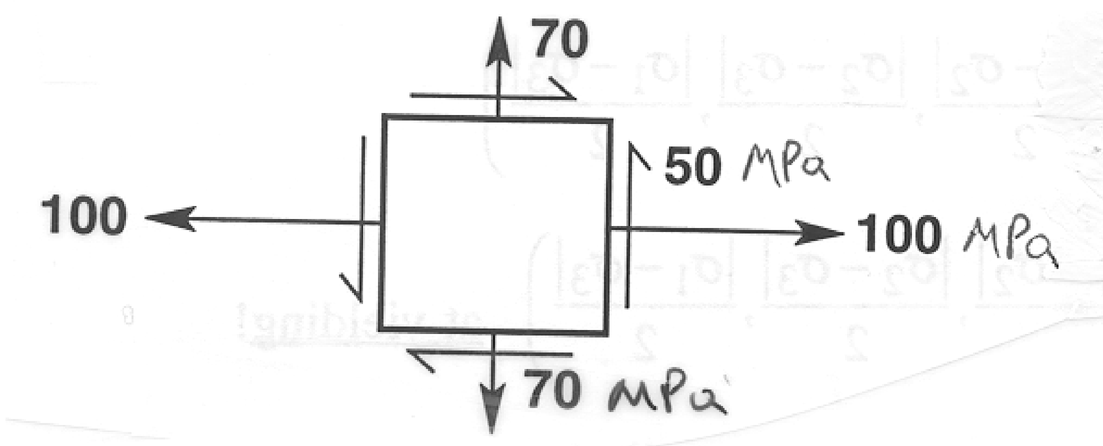

Figure 7: A plane stress state.

We construct a Mohr circle to evaluate the principal stresses on the plane. From the construct shown in Fig. 8 we conclude that:

$$R = \sqrt{(\frac{100-70}{2})^2 + 50^2} = 52 \text{ MPa} \tag{22}$$

$$C = \frac{\sigma_x + \sigma_y}{2} = 85 \text{ MPa} \tag{23}$$

From the above, we conclude that: $\sigma_1 = 137$ MPa and $\sigma_2 = 33$ MPa. This together with $\sigma_3 = 0$ (plane stress) gives that: $\sigma_{\max} = 137$ Mpa and $\sigma_{\min} = 0$. Substitution into the Tresca criterion results in the following:

$$\tau_{\max} = \frac{137-0}{2} = \frac{Y}{2} \tag{24}$$

from which we conclude that $Y = 137$ MPa.

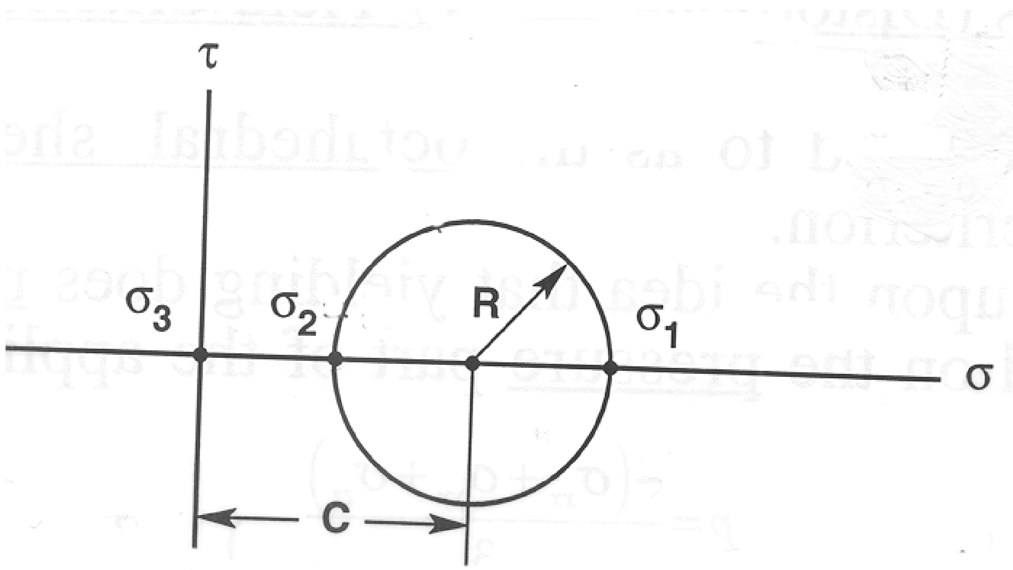

Figure 8: A Mohr circle construction for the plane stress case of the Fig. 7 above.

The von-Mises yield criterion

Define an equivalent stress $\bar{\sigma}_{\mathrm{VM}}$ as follows:

$$\bar{\sigma}_{\mathrm{VM}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \tag{25}$$

(in terms of principal stress components) or as

$$\bar{\sigma}_{\mathrm{VM}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)} \tag{26}$$

(in terms of the stress components in the x, y, z coordinate system).

Note that the two equations above are equivalent (i.e. the right hand side of Equation (26) is invariant – does not change as we 'rotate' the stress components from one coordinate system to another).

According to the von-Mises yield criterion, in a general multi-dimensional stress state, yielding occurs when the von-Mises equivalent stress becomes equal to the yield stress Y in tension, i.e. yield occurs when:

$$\text{von}-\text{Mises yield criterion}: \;\; \text{Yielding occurs when}: \quad \bar{\sigma}_{\mathrm{VM}} = Y \tag{27}$$

As expected (and by an obvious design of the von-Mises criterion), for uniaxial tension ($\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0$) the von-Mises yield criterion predicts that yielding occurs when:

$$\bar{\sigma}_{\mathrm{VM}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-0)^2+(0-0)^2+(0-\sigma_1)^2} = \sigma_1 = Y \tag{28}$$

or when

$$\sigma_1 = Y \tag{29}$$

Any different answer was going to be not-acceptable!!!

Note: Recall that from lecture 4 (Equations 21 and 22), the distortional energy calculated for 'elastic deformations' was given as follows:

$$W_d = \frac{1}{12G}[(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)] \tag{30}$$

from which using the earlier given expressions for $\bar{\sigma}_{\mathrm{VM}}$, we can write:

$$W_d = \frac{1}{6G}\,\bar{\sigma}_{\mathrm{VM}}^2, \quad \text{or} \quad \bar{\sigma}_{\mathrm{VM}} = \sqrt{6\;G\;W_d} \tag{31}$$

It now becomes clear that one can state the von-Mises yield criterion in an equivalent form as follows:

$$\text{Equivalent form of von} - \text{Mises criterion}: \ \text{Yielding occurs when}: \quad W_d = \frac{1}{6G}\, Y^2 \qquad (32)$$

This form of the von-Mises yield criterion justifies the alternative name of the von-Mises criterion as the 'Maximum distortion energy yield criterion'. This form of interpretation will not be further used in this course as the presentation using an equivalent stress is much easier to interpret and use in the calculations.

Examples of the application of the von-Mises yield criterion

Example 1: Equi-biaxial tension $\sigma_1 = \sigma_2 = \sigma$ and $\sigma_3 = 0$ (Fig. 3)

Substitution in the expression for $\bar{\sigma}_{\mathrm{VM}}$ gives:

$$\bar{\sigma}_{\mathrm{VM}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} = \frac{1}{\sqrt{2}}\sqrt{(\sigma-\sigma)^2+(\sigma-0)^2+(0-\sigma)^2} = \sigma \qquad (33)$$

So for this case yielding occurs when $\sigma = Y$.

Example 2: Hydrostatic pressure: $\sigma_1 = \sigma_2 = \sigma_3 = -p$ (Fig. 4)

Substitution in the expression for $\bar{\sigma}_{\mathrm{VM}}$ gives:

$$\begin{aligned} \bar{\sigma}_{\mathrm{VM}} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{(-p-(-p))^2+(-p-(-p))^2+(-p-(-p))^2} = 0 \neq Y \end{aligned} \qquad (34)$$

So as it was the case with the Tresca criterion, yielding does not occur in the von-Mises criterion for the case of hydrostatic pressure.

Example 3: Thin pressurized tube with end caps ($\frac{r}{t} > 10$) (Fig. 5)

Substitution in the expression for $\bar{\sigma}_{\mathrm{VM}}$ of $\sigma_r = 0$, $\sigma_\theta = \frac{pr}{t}$ and $\sigma_z = \frac{pr}{2t}$ gives the following:

$$\begin{aligned} \bar{\sigma}_{\mathrm{VM}} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{(0-\frac{pr}{t})^2+(\frac{pr}{t}-\frac{pr}{2t})^2+(\frac{pr}{2t}-0)^2} = \frac{\sqrt{3}}{2}\frac{pr}{t} \end{aligned} \qquad (35)$$

So yielding occurs when:

$$\frac{\sqrt{3}}{2}\frac{pr}{t} = Y, \qquad \text{or} \quad p = \frac{2}{\sqrt{3}}\, Y \frac{t}{r} \qquad (36)$$

Example 4: Pure shear: $\sigma_1 = -\sigma_2 = \tau$ and $\sigma_3 = 0$ (Fig. 6)

Substitution in the expression for $\bar{\sigma}_{\text{VM}}$ of $\sigma_1 = \tau$, $\sigma_2 = -\tau$ and $\sigma_3 = 0$ gives the following:

$$\begin{aligned} \bar{\sigma}_{\text{VM}} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{(\tau - (-\tau))^2 + (-\tau - 0)^2 + (0 - \tau)^2} = \sqrt{3}\tau \end{aligned} \tag{37}$$

So in pure shear, yielding occurs according to the von-Mises criterion when:

$$\sqrt{3}\tau = Y, \quad \text{or} \quad \tau \equiv \kappa = \frac{Y}{\sqrt{3}} \tag{38}$$

Note that for the Tresca criterion we computed that $\kappa = \frac{Y}{2}$!!

The von-Mises criterion for the general plane stress state: $\sigma_2 = 0$

Let us apply the von-Mises criterion in the general case of plane stress conditions: $\sigma_2 = 0$. Substitution into the expression for $\bar{\sigma}_{\text{VM}}$ gives the following:

$$\begin{aligned} \bar{\sigma}_{\text{VM}} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - 0)^2 + (0 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2} \end{aligned} \tag{39}$$

Thus according to the von-Mises yield criterion, yielding occurs when:

$$\text{For plane stress } (\sigma_2 = 0) \text{ yielding occurs when : } \sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2 = Y^2 \tag{40}$$

If you plot the above equation on the $\sigma_1 - \sigma_3$ plane, you obtain an ellipse (Fig. 9). Note that the Tresca and von-Mises criteria agree in tension but differ by about 15% in shear (Recall the corresponding values of κ: $\frac{Y}{2}$ and $\frac{Y}{\sqrt{3}}$).

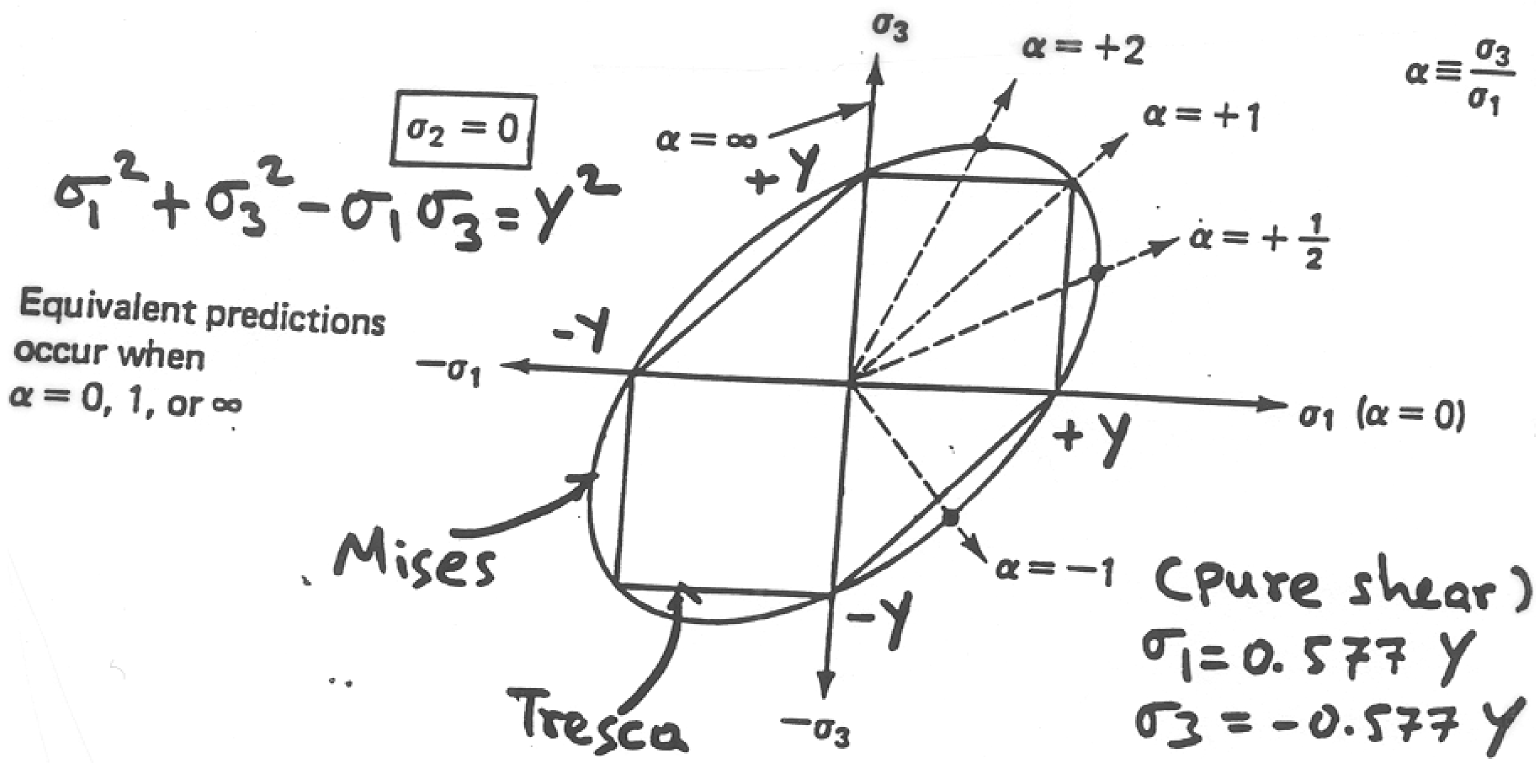

Figure 9: The Tresca and von-Mises yield loci for the same value of Y showing certain loading paths (i.e. for varying $\alpha = \frac{\sigma_3}{\sigma_1}$).