MAE 212: Spring 2001
Lecture 10

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PLASTIC INCOMPRESSIBILITY, FLOW RULE AND YIELDING IN PLANE STRAIN & AXISYMMETRIC PROBLEMS

Plastic incompressibility and Flow Rule

In order to motivate the stress/strain relations in the plastic regime, let us consider again the tensile test. The specimen is loaded in the 1-direction (see Fig. 1). In addition to straining in the 1-direction, shrinking will develop in the 2 and 3 directions.

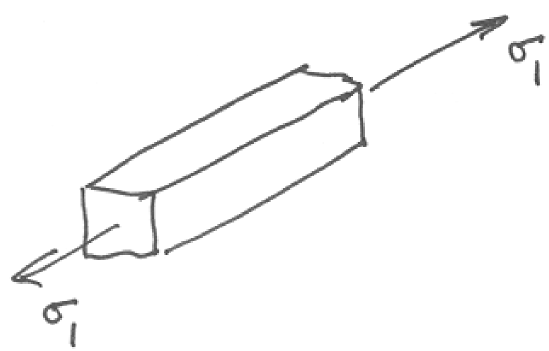

Figure 1: The tensile test.

For isotropic deformations:

$$\epsilon_2 = \epsilon_3 \tag{1}$$

Note that here ϵ_1, ϵ_2 and ϵ_3 are true strains and elastic deformation is neglected as discussed in an earlier lecture (i.e. $\epsilon_1 \approx \epsilon_1^p$, $\epsilon_2 \approx \epsilon_2^p$ and $\epsilon_3 \approx \epsilon_3^p$).

Experimental evidence points to the fact that plastic deformations are incompressible ($\frac{\Delta V}{V} = 0$):

$$\frac{\Delta V}{V} = \epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \tag{2}$$

From equations (1) and (2) we conclude that for the tensile test:

$$\epsilon_2 = \epsilon_3 = -\frac{\epsilon_1}{2} \tag{3}$$

Note: If the deformation was elastic, we can write using isotropy and the definition of Poisson's ratio:

$$\epsilon_2 = \epsilon_3 = -\nu \epsilon_1 \tag{4}$$

Many times students (and authors of some textbooks!) confuse the significance of equation (3) and think that it is derived from equation (4) using $\nu = 0.5$ (recall that for $\nu = 0.5$, elastic deformations are incompressible – see Equ. (8) of lecture 4). The bottom line is that equation (4) is not applicable here because we only consider plastic deformations and elasticity is neglected.

Continuing with the uniaxial tensile test (Fig. 1), let us calculate the deviatoric stress components:

$$\begin{aligned} \sigma_1' &= \sigma_1 - \sigma_m = \sigma_1 - \frac{\sigma_1 + 0 + 0}{3} = \frac{2\sigma_1}{3} \\ \sigma_2' &= \sigma_2 - \sigma_m = 0 - \frac{\sigma_1 + 0 + 0}{3} = -\frac{\sigma_1}{3} \\ \sigma_3' &= \sigma_3 - \sigma_m = 0 - \frac{\sigma_1 + 0 + 0}{3} = -\frac{\sigma_1}{3} \end{aligned} \tag{5}$$

Recall that the deviatoric stress components are defined by removing the hydrostatic stress from the normal stress components. Since plastic deformations are incompressible they depend on the deviatoric stress components and not on the hydrostatic pressure.

Using equations (3) and (5), we can write:

$$\frac{\epsilon_2}{\sigma_2'} = \frac{\epsilon_3}{\sigma_3'} = \frac{-\frac{\epsilon_1}{2}}{-\frac{\sigma_1}{3}} = \frac{\epsilon_1}{\frac{2\sigma_1}{3}} = \frac{\epsilon_1}{\sigma_1'} \tag{6}$$

or

$$\frac{\epsilon_1}{\sigma_1'} = \frac{\epsilon_2}{\sigma_2'} = \frac{\epsilon_3}{\sigma_3'} = \text{constant} \tag{7}$$

Equation (7) is called the **flow rule** (the stress/strain relation in the plastic region). The flow rule plays 'short of' the role that Hooke's law plays in the elastic region. Both the flow rule (Eq. (7)) and incompressibility condition (Eq. (2)) are valid for multi-dimensional deformations as well even though the above discussion was restricted to the uniaxial tensile test.

Note: A word on the history dependence of plastic deformations

In elastic deformations, the stress at a given level of strain depends only from that level of strain (e.g. $\sigma = E\epsilon$) and not how we reached that strain. This is not the case in plastic deformations where the stress at a given strain depends on the history of deformation that brought you from zero strain to strain ϵ.

For this reason, we work with strain increments as we deform the material. We calculate the plastic strain by adding these increments:

$$\epsilon = \int_{\text{over the path (history) of deformation}} d\epsilon \tag{8}$$

The correct form of the flow rule to be used from now on is thus the following:

$$\frac{d\epsilon_1}{\sigma_1'} = \frac{d\epsilon_2}{\sigma_2'} = \frac{d\epsilon_3}{\sigma_3'} = d\lambda \ \ (= \text{material constant}) \tag{9}$$

and the correct form of the incompressibility condition is:

$$d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0 \tag{10}$$

We emphasize again that even though **the above equations** were motivated using the tensile test, they **are valid for multi-dimensional deformations** as well.

The flow rule (Equ. (9)) is re-written here in principal stress (strain) axes as follows:

$$\begin{aligned} d\epsilon_1 &= d\lambda \ \sigma_1' \\ d\epsilon_2 &= d\lambda \ \sigma_2' \\ d\epsilon_3 &= d\lambda \ \sigma_3' \end{aligned} \tag{11}$$

where $d\lambda$ is a material parameter to be calculated (but not in MAE212!).

Recall that the deviatoric (principal) stress components σ_i', i=1,2,3 are defined as:

$$\begin{aligned} \sigma_1' &= \sigma_1 - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{2\sigma_1 - (\sigma_2 + \sigma_3)}{3} \\ \sigma_2' &= \sigma_2 - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{2\sigma_2 - (\sigma_1 + \sigma_3)}{3} \\ \sigma_3' &= \sigma_3 - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{2\sigma_3 - (\sigma_1 + \sigma_2)}{3} \end{aligned} \tag{12}$$

You notice that each incremental strain increment is proportional to the corresponding deviatoric stress component (with the constant of proportionality being the same for all components $1, 2, 3$). Also we emphasize once more that plastic strain increments depend only through the deviatoric stress components and not through the hydrostatic stress σ_m.

Plastic work & Effective Strain for the von-Mises yield criterion

In lecture 4 (Equ. (15)), we introduced the general form of the incremental work per unit volume for one-dimensional deformations:

$$dw = \sigma_{xx} d\epsilon_{xx} \tag{13}$$

We can generalize the above expression in multi-dimensional deformations as follows:

$$dw = \sigma_{xx} d\epsilon_{xx} + \sigma_{yy} d\epsilon_{yy} + \sigma_{zz} d\epsilon_{zz} + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} \tag{14}$$

The above expressions are general and valid for both elastic and plastic deformations. However, here we are again concerned with negligible elastic deformations and large plastic strains.

Recall that for von-Mises yielding, we already have defined the equivalent stress $\bar{\sigma}_{VM}$ as follows:

$$\bar{\sigma}_{VM} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)} \tag{15}$$

It would be nice if we can define an equivalent (or effective) strain increment $d\bar{\epsilon}$. We here define an effective $d\bar{\epsilon}$ that is work conjugate to $\bar{\sigma}_{VM}$, i.e. such that:

$$dw = \sigma_{xx} d\epsilon_{xx} + \sigma_{yy} d\epsilon_{yy} + \sigma_{zz} d\epsilon_{zz} + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} \equiv \bar{\sigma}_{VM} d\bar{\epsilon} \tag{16}$$

The idea here is simple: Once you define an equivalent stress (here $\bar{\sigma}_{VM}$), then $d\bar{\epsilon}$ cannot be defined arbitrarily but it must obey the work-conjugate relation of Equ. (16).

It can be shown using equations (15), (16) and equations (9), (10) that in terms of principal strain increments:[1]

$$d\bar{\epsilon} = \sqrt{\frac{2}{3}(d\epsilon_1^2 + d\epsilon_2^2 + d\epsilon_3^2)} \tag{17}$$

Note that very often in this course you can simplify the expression on the right hand side of Equ. (17) using the incompressibility condition $d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0$.

Note: Verify using Equ. (3) that for a uniaxial tensile test, Equ. (17) predicts that $d\bar{\epsilon} = d\epsilon_1$ as it should be!!

Note: Equation (17) is in terms of principal strain components. We will not need in this course the expression in terms of strain components in the general x, y, z coordinate system (recall, anyways, that we have not defined and we do not plan to define in MAE212 true shear strains!!)

Note: In this course, we will not need to define $d\bar{\epsilon}$ for the Tresca criterion. However, just keep in mind that equation (17) is only good for the von-Mises yield criterion and provides

[1] You do not have to worry about the proof of this equation!

the equivalent strain increment that is work conjugate to the von-Mises equivalent stress $\bar{\sigma}_{VM}$.

We will next apply the incompressibility condition, flow rule, equivalent stress & equivalent strain definitions to a number of examples that will be useful in the analysis of forming processes (such as forging, extrusion, rolling, etc.). Even though the derivations given here may look of no relevance to anything, you should trust us that using these equations we will be able later in the course to derive many practical results in forming process analysis and design.

Simplified expressions $\bar{\sigma}_{VM}$, $d\bar{\epsilon}$, yield condition & plastic work

for plane strain problems

Assume that $1, 2, 3$ are the principal strain axes and that $\epsilon_2 = 0$ (plane strain on the 13 plane). From the flow rule (Equ. (11)) using the expressions of Eq. (12) for the deviatoric stress components, we compute:

$$d\epsilon_2 = d\lambda\ \sigma_2' = 0, \quad ---> \quad \sigma_2' = \frac{2\sigma_2 - (\sigma_1 + \sigma_3)}{3} = 0, \quad --> \quad \sigma_2 = \frac{\sigma_1 + \sigma_3}{2} \tag{18}$$

Using the above equation, one can simplify $\bar{\sigma}_{VM}$ as follows:

$$\begin{aligned}
\bar{\sigma}_{VM} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \frac{\sigma_1 + \sigma_3}{2})^2 + (\frac{\sigma_1 + \sigma_3}{2} - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{1}{4}(\sigma_1 - \sigma_3)^2 + \frac{1}{4}(\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} \\
&= \frac{\sqrt{3}}{2}|\sigma_1 - \sigma_3|
\end{aligned} \tag{19}$$

Using this important equation, we can now write the von-Mises yield condition for plane strain as follows:

$$\text{Plane strain } (\epsilon_2 = 0) \text{ Mises yield condition} : \bar{\sigma}_{VM} = \frac{\sqrt{3}}{2}|\sigma_1 - \sigma_3| = Y, \quad ---> \quad |\sigma_1 - \sigma_3| = \frac{2}{\sqrt{3}}Y \tag{20}$$

Let us now calculate $d\bar{\epsilon}$ for this plane strain condition. Using $\epsilon_2 = 0$, the incompressibility condition (Eq. (10)) results in the following:

$$d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0, \quad ---> \quad d\epsilon_1 + 0 + d\epsilon_3 = 0, \quad ---> \quad d\epsilon_3 = -d\epsilon_1 \tag{21}$$

Substitution of this equation into Eq. (17) results in the following:

$$d\bar{\epsilon} = \sqrt{\frac{2}{3}(d\epsilon_1^2 + d\epsilon_2^2 + d\epsilon_3^2)} = \sqrt{\frac{2}{3}(d\epsilon_1^2 + 0 + (-d\epsilon_1)^2)} = \frac{2}{\sqrt{3}}|d\epsilon_1| = \frac{2}{\sqrt{3}}|d\epsilon_3| \qquad (22)$$

Using the above expression for $d\bar{\epsilon}$ and the von-Mises yield condition, we can compute the incremental work per unit volume as follows:

$$dw = \bar{\sigma}_{VM} d\bar{\epsilon} = Y\frac{2}{\sqrt{3}}|d\epsilon_1| = Y\frac{2}{\sqrt{3}}|d\epsilon_3| \qquad (23)$$

The above expression will be very useful in energy based analysis of forming processes later in this course.

<u>An example of a plane strain drawing process</u>

As an application of the above equations, let us consider the plane strain drawing process shown in Fig. 2. We here assume $\epsilon_z = 0$ and that x, y, z are principal axes. Following

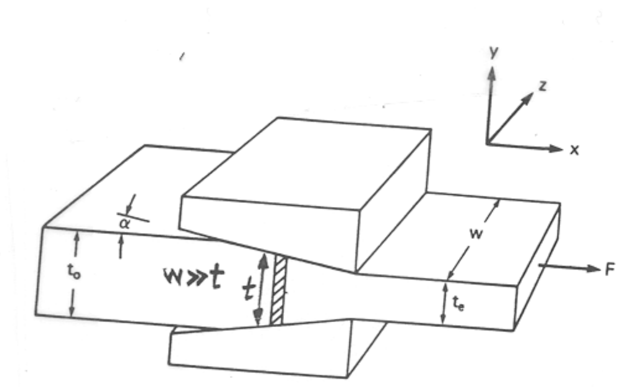

Figure 2: A drawing process. Because $w >> t$, we can approximate that $\epsilon_z = 0$ and thus assume plane strain conditions. The material yields inside the deformation zone (from the entrance to the die to the exit from the die. We assume that the axes x, y, z remain principal axes everywhere inside the deformation zone.

the results given earlier for general plane strain conditions, we can summarize the following results for this plane strain drawing process (note that here $\epsilon_z = 0$, whereas in the general case examined we had $\epsilon_2 = 0$):

$$\begin{aligned}
\text{Equivalent Stress}: \ \bar{\sigma}_{VM} &= \frac{\sqrt{3}}{2}|\sigma_x - \sigma_y| \\
\text{Yield Condition}: \ |\sigma_x - \sigma_y| &= \frac{2}{\sqrt{3}}Y \\
\text{Equivalent Strain Increment}: \ d\bar{\epsilon} &= \frac{2}{\sqrt{3}}|d\epsilon_y| \\
\text{Incremental Plastic Work}: \ dw &= Y\frac{2}{\sqrt{3}}|d\epsilon_y| \qquad (24)
\end{aligned}$$

Let $p > 0$ be the pressure at the contact interface between the die and the workpiece. Assuming that the semi-angle α (Fig. 2) of the die is very small (i.e. small reductions), we can consider that the axis y is approximately normal to the die/workpiece contact interface and approximate that: $\sigma_y = -p$. Noting that $\sigma_x > 0$, we can re-write equations (24(a,b)) as follows:

$$\begin{aligned} \text{Equivalent Stress}: \ \bar{\sigma}_{VM} &= \frac{\sqrt{3}}{2}(\sigma_x + p) \\ \text{Yield Condition}: \ \sigma_x + p &= \frac{2}{\sqrt{3}}Y \end{aligned} \tag{25}$$

Note that equations (24) and (25) are valid everywhere inside the deformation zone.

An example of a plane strain forging process that was examined in Lab Module I

Let us condider the plane strain forging process shown in Fig. 3 examined in Module I of the laboratory experiments. The workpiece is constrained in direction 2 ($\epsilon_2 = 0$) and in addition the surface with normal axis 3 is free ($\sigma_3 = 0$). Using our earlier results for plane strain, we can summarize the following equations that are very useful in Module I of the lab as well as for further analysis of the particular forging processes.

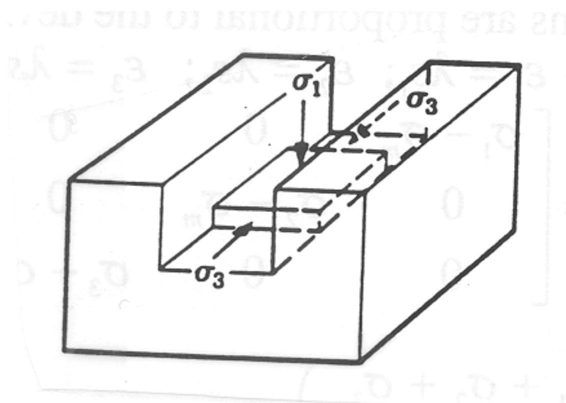

Figure 3: A plane strain compression test as examined in Module I of the laboratory experiments. The workpiece is constrained in direction 2 and it is free to expand in direction 3.

$$\begin{aligned} \text{Stress Constraint}: \ \sigma_2 &= \frac{\sigma_1}{2} \\ \text{Equivalent Stress}: \ \bar{\sigma}_{VM} &= \frac{\sqrt{3}}{2}|\sigma_1| \\ \text{Yield Condition}: \ |\sigma_1| &= \frac{2}{\sqrt{3}}Y \end{aligned}$$

$$\text{Equivalent Strain Increment}: \; d\bar{\epsilon} \;=\; \frac{2}{\sqrt{3}}|d\epsilon_1|$$
$$\text{Incremental Plastic Work}: \; dw \;=\; Y\frac{2}{\sqrt{3}}|d\epsilon_1| \qquad (26)$$

Simplified expressions $\bar{\sigma}_{VM}$, $d\bar{\epsilon}$, yield condition & plastic work for axially symmetric problems

An axisymmetric body is assumed to have symmetry around the z-axis and the deformation/stresses have no dependence on the coordinate θ. In this course we assume that the r, θ, z axes are principal axes (of stress or strain). So all shear stresses shown in Fig. 4 are zero. Typical cases that we will approximate as axisymmetric include the deformation of cylinders with circular section, extrusion/drawing of rods, etc.

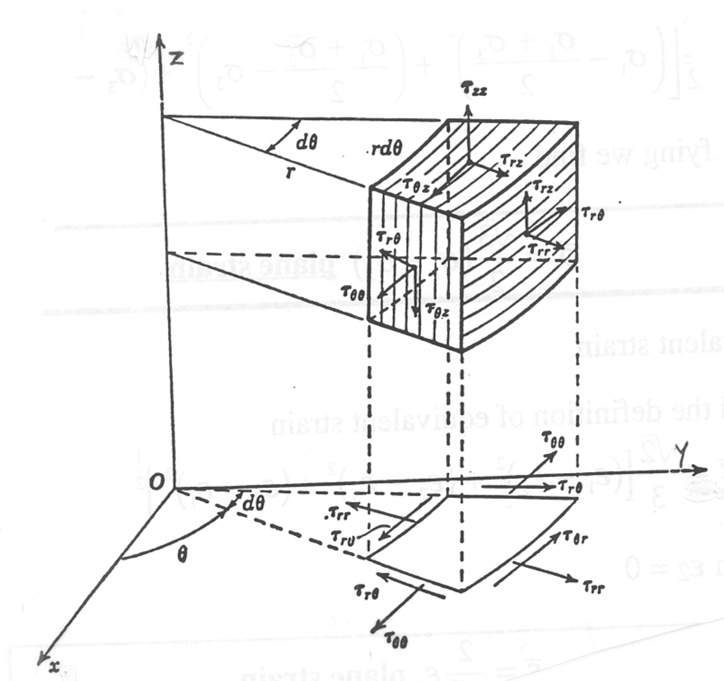

Figure 4: A reminder of the stress components in a polar coordinate system. In an axisymmetric problem, we assume that the axes r, θ, z are principal and that there is no θ dependence of the stresses or strains.

For reasons that are not necessary to justify in this course, we will approximate that during axisymmetric deformations, the following conditions are true:

$$\epsilon_r \;=\; \epsilon_\theta$$
$$\sigma_r \;=\; \sigma_\theta \qquad (27)$$

These equations are not general, but we will accept them here in order to allow ourselves to work with circular geometry, etc.

Let us simplify $\bar{\sigma}_{VM}$ for axisymmetric bodies:

$$\begin{aligned}
\bar{\sigma}_{VM} &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{(\sigma_r-\sigma_r)^2+(\sigma_r-\sigma_z)^2+(\sigma_z-\sigma_r)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{0+(\sigma_r-\sigma_z)^2+(\sigma_z-\sigma_r)^2} \\
&= |\sigma_r-\sigma_z| \qquad (28)
\end{aligned}$$

and thus the yield condition for axisymmetric deformations can be written as follows:

$$\text{Yield Condition For Axisymmetric Bodies}: \quad |\sigma_r-\sigma_z| = Y \qquad (29)$$

Using Equ. (27(a)) and the incompressibility condition $d\epsilon_r + d\epsilon_\theta + d\epsilon_z = 0$, we can see that:

$$d\epsilon_r = d\epsilon_\theta = -\frac{d\epsilon_z}{2} \qquad (30)$$

The above equation can be used to simplify the expression for $d\bar{\epsilon}$ as follows:

$$d\bar{\epsilon} = \sqrt{\frac{2}{3}(d\epsilon_r^2+d\epsilon_\theta^2+d\epsilon_z^2)} = \sqrt{\frac{2}{3}((-\frac{d\epsilon_z}{2})^2+(-\frac{d\epsilon_z}{2})^2+d\epsilon_z^2)} = |d\epsilon_z| \qquad (31)$$

In summary, for axisymmetric deformations the following useful results are obtained:

$$\begin{aligned}
\text{Stress Constraint}: \ \sigma_r &= \sigma_\theta \\
\text{Strain Constraint}: \ \epsilon_r &= \epsilon_\theta \\
\text{Equivalent Stress}: \ \bar{\sigma}_{VM} &= |\sigma_r-\sigma_z| \\
\text{Yield Condition}: \ |\sigma_r-\sigma_z| &= Y \\
\text{Equivalent Strain Increment}: \ d\bar{\epsilon} &= |d\epsilon_z| \\
\text{Incremental Plastic Work}: \ dw &= Y|d\epsilon_z| \qquad (32)
\end{aligned}$$

Conditions for continuous (sustained) yielding:

All forms of yield conditions we have seen up to now are for initiation of yielding (yield stress = Y). What do we suppose to do for continuing straining (work-hardening)?

To make things simple, let us consider power-law hardening ($\sigma = K\epsilon^n$) and concentrate on the von-Mises yield condition. To define the condition between the stress components in order to sustain yielding as the material hardens, we proceed as follows:

- (A) Using the effective strain $\bar{\epsilon}$, we calculate the yield stress (flow stress) of the material as: $K\bar{\epsilon}^n$
- (B) The von-Mises criterion is now modified to take the form:

$$\bar{\sigma}_{VM} = K\bar{\epsilon}^n \tag{33}$$

i.e. at each level of $\bar{\epsilon}$, the effective von-Mises stress is equal to the current yield stress which is calculated based on the uniaxial hardening law but using as strain the equivalent strain $\bar{\epsilon}$.

As such all expressions given earlier for the yield condition or the work expressions are applicable for sustained yielding by using the current yield stress ($K\bar{\epsilon}^n$) instead of Y.

As an example, for the case of plane drawing of Fig. 2 with a power law hardening material model and an initial thickness t_0, you should be able to easily show that at an arbitrary location inside the deformation zone where the thickness is t, the equivalent strain increment, yield stress, yield condition and incremental work per unit volume can be written as follows:

$$\begin{aligned}
\text{Equivalent Strain Increment}: \ d\bar{\epsilon} &= \frac{2}{\sqrt{3}}|d\epsilon_y| \\
\text{Yield Stress}: \ Y &= K(\frac{2}{\sqrt{3}}ln\frac{t_0}{t})^n \\
\text{Yield Condition}: \ \sigma_x + p &= \frac{2}{\sqrt{3}}K(\frac{2}{\sqrt{3}}ln\frac{t_0}{t})^n \\
\text{Incremental Plastic Work}: \ dw &= Y\frac{2}{\sqrt{3}}|d\epsilon_y|
\end{aligned} \tag{34}$$

In these equations, we assume that the stresses/strains, yield stress, etc. are only functions of x (i.e. the same in each cross-section of the workpiece) and that the material is yielding everywhere within the deformation zone. The restrictions of small reductions, small die semi-angles, etc. discussed earlier are applicable here as well.