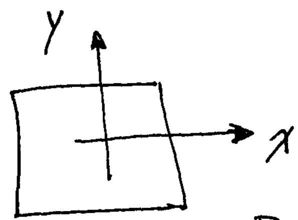
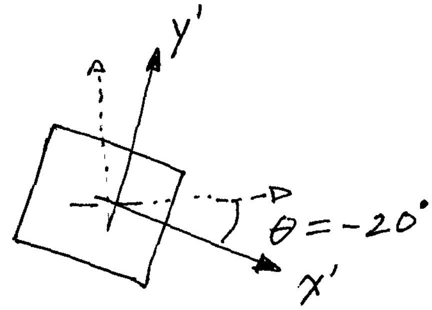
MAE 212 – SPRING 2001
HW2 SOLUTIONS

1





$$\sigma_y = \frac{P}{A} = \frac{P}{6\,in^2}$$

$$\sigma_x = \tau_{xy} = 0$$

$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

$$= \frac{P}{12\,in^2} - \left(-\frac{P}{12\,in^2}\right)\cos(-40^\circ) - 0 \approx 0.147\,\frac{P}{in^2}$$

$$\tau_{x'y'} = -\frac{\sigma_x-\sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$= -\left(-\frac{P}{12\,in^2}\right)\sin(-40^\circ) + 0 = -0.0536\,\frac{P}{in^2}$$

For allowable stresses, we must have:

$$\sigma_y' \le 95\,psi \implies P \le \frac{95\,psi}{0.147/in^2} = 646\,lb$$

$$|\tau_{x'y'}| \le 85\,psi \implies P \le \frac{85\,psi}{0.0536/in^2} = 1586\,lb$$

Thus, largest allowable load $\boxed{P = 646\,lb}$

2 Using the equation

$$e_{x'x'} = \frac{e_{xx}+e_{yy}}{2} + \frac{e_{xx}-e_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (1)$$

for $\theta = 0$, $60°$ and $120°$, gives the following:

$$e_a = e_x$$

$$e_b = \frac{1}{4}e_x + \frac{3}{4}e_y + \frac{\sqrt{3}}{4}\gamma_{xy}$$

$$e_c = \frac{1}{4}e_x + \frac{3}{4}e_y - \frac{\sqrt{3}}{4}\gamma_{xy}$$

From these,

$$\left.\begin{aligned} e_x &= e_a \\ e_y &= \frac{1}{3}\left[2(e_b+e_c)-e_a\right] \\ \gamma_{xy} &= \frac{2}{\sqrt{3}}(e_b-e_c) \end{aligned}\right\} \qquad (2)$$

Note that the relationships between e_a, e_b and e_c may be observed from a Mohr's circle construction corresponding to the state e_x, e_y and γ_{xy} at the point under consideration.

UPON SUBSTITUTION OF NUMERICAL VALUES,

$$e_x = 200\,\mu$$

$$e_y = -133\,\mu$$

$$\gamma_{xy} = 462\,\mu$$

AND THE PRINCIPAL STRAINS MAY BE OBTAINED FROM

$$e_{1,2} = \frac{e_x + e_y}{2} \pm \sqrt{\left(\frac{e_x - e_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (3)$$

SUBSTITUTING NUMERICAL VALUES GIVES:

$$e_{1,2} = \frac{200 - 133}{2} \pm \sqrt{\left(\frac{200+133}{2}\right)^2 + \left(\frac{462}{2}\right)^2}$$

$$= 33.33 \pm 284.75 \Rightarrow e_1 = 318\,\mu$$

$$e_2 = -254\,\mu$$

THE MAX SHEAR STRAIN IS FOUND FROM

$$\gamma_{max} = \pm (e_1 - e_2) = \pm 572\,\mu$$

IN THE PLANE OF THE ROSETTE

NOW THE ORIENTATION OF THE PRINCIPAL AXES IN THE PLANE OF THE ROSETTE ARE GIVEN BY

$$\tan(2\theta_p) = \frac{\gamma_{xy}}{e_x - e_y} \qquad (4)$$

$$2\theta_p = \tan^{-1}\left(\frac{462}{333}\right)$$

SO THAT

$$\theta_p' = 27.1^\circ$$
$$\theta_p'' = 117.1^\circ$$

WHEN θ_p' IS SUBSTITUTED INTO EQN (1) TOGETHER WITH THE VALUES OF e_x, e_y, γ_{xy}, WE OBTAIN 318μ. THEREFORE, 27.1° AND 117.1° ARE THE RESPECTIVE DIRECTIONS OF e_1 AND e_2, MEASURED CCW FROM THE HORIZONTAL AXIS. NOTE - SOLUTION NOT COMPLETE WITHOUT THIS STATEMENT.

THE <u>PRINCIPAL STRESSES</u> MAY NOW BE FOUND FROM THE GENERALIZED HOOKE'S LAW.

$$\left.\begin{aligned} e_1 &= \frac{1}{E}(\sigma_1 - \nu(\sigma_2+\sigma_3)) \\ e_2 &= \frac{1}{E}(\sigma_2 - \nu(\sigma_1+\sigma_3)) \end{aligned}\right\}$$

FOR $\sigma_3 = \sigma_z = 0$ GET.

$$\begin{aligned} e_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2) \\ e_2 &= \frac{1}{E}(\sigma_2 - \nu\sigma_1) \end{aligned} \Rightarrow \begin{aligned} \sigma_1 &= \frac{E}{1-\nu^2}(e_1 + \nu e_2) \\ \sigma_2 &= \frac{E}{1-\nu^2}(e_2 + \nu e_1) \end{aligned}$$

$$\sigma_1 = \frac{250 \times 10^9}{(1-0.16)}\left[318 + 0.4(-254)\right]10^{-6} = 64.41\ \mathrm{MPa}$$

$$\sigma_2 = \frac{250 \times 10^9}{(1-0.16)}\left[-254 + 0.4(318)\right]10^{-6} = -37.74\ \mathrm{MPa}$$

THE DIRECTIONS OF σ_1, σ_2 ARE THE SAME AS THE DIRECTIONS OF e_1, e_2 (NOTE – IN A COORDINATE SYSTEM IN WHICH THERE'S NO SHEAR STRAIN, THE GENERALIZED HOOKE'S LAW GIVES NO SHEAR STRESS)

THE <u>MAX SHEAR</u> STRESS IS:

$$\tau_{max} = G\,\gamma_{max}$$

$$= \frac{E}{2(1+\nu)}\gamma_{max}$$

$$= \frac{250 \times 10^9}{2(1+0.4)}\,572 \times 10^{-6} = 51.1\ \mathrm{MPa}$$

NOTE: AS A CHECK, $\frac{\sigma_1 - \sigma_2}{2}$ YIELDS THE SAME RESULT FOR τ_{max}.

THE OUT OF PLANE NORMAL STRAIN e_z CAN BE CALCULATED USING HOOKE'S GENERALIZED LAW.

$$e_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \quad \xrightarrow[\sigma_z = 0]{\text{FOR}} \quad e_x + e_y = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$e_y = \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \qquad\qquad + \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$e_x + e_y = \frac{1-\nu}{E}(\sigma_x + \sigma_y)$$

$$\sigma_x + \sigma_y = \frac{E}{1-\nu}(e_x + e_y)$$

$$\text{BUT} \quad e_z = \frac{1}{E}(\cancelto{0}{\sigma_z} - \nu(\sigma_x + \sigma_y))$$

$$= -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{\nu}{1-\nu}(e_x + e_y)$$

SUBSTITUTING NUMBERS GIVES

$$e_z = -\frac{0.4}{1-0.4}(200-133) = -44.7\,\mu$$

NOW, AT THE FREE SURFACE ON WHICH THE STRAIN ROSETTE IS APPLIED, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ SO. THAT BY THE GENERALIZED HOOKE'S LAW $\gamma_{xz} = \gamma_{yz} = 0$.

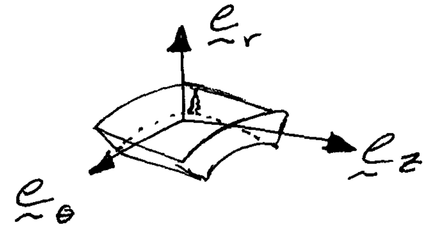
THUS e_z, e_1 AND e_2 ARE THE PRINCIPAL STRAINS.

BECAUSE e_z HAS A VALUE BETWEEN e_1 AND e_2, THE TRUE MAX SHEARING STRAIN IS

$$(\gamma_{max})_t = \pm(e_{max} - e_{min}) = \pm 572\,\mu$$

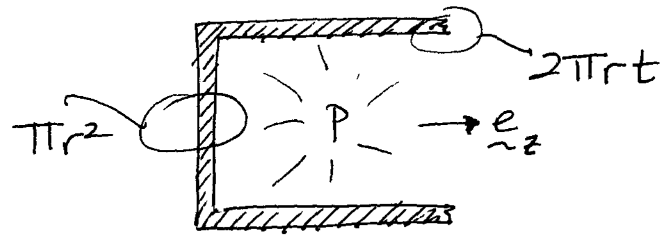
AS DONE EARLIER

3 First, obtain expressions for the in-plane stresses.

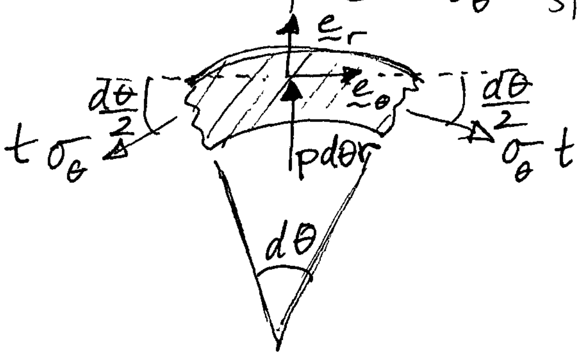


There are no shear stresses in the coordinate system aligned with the pressure vessel axes.

Along the e_z direction, $F = \pi r^2 p$ from the ends of the vessel and the load is carried over an area $A \approx 2\pi r t$. Thus $\boxed{\sigma_z = \frac{\pi r^2 p}{2\pi r t} = \frac{pr}{2t}}$



For the σ_θ stress component, or hoop stress



$$\left(\Sigma F\right) \cdot e_\theta = 0 = p\,d\theta\,r - 2\left(\frac{d\theta}{2}\sigma_\theta t\right)$$

$$\Rightarrow \sigma_\theta = \frac{pr}{t}$$

using $\sin\left(\frac{d\theta}{2}\right) \approx \frac{d\theta}{2}$

Note that σ_r is on the order of p and is neglected in comparison to σ_θ and σ_z.

The maximum shear stress in the θz plane is:

$$\tau_{max}\Big|_{\theta z \text{ plane}} = \frac{\sigma_\theta - \sigma_z}{2} = \frac{\frac{pr}{t} - \frac{pr}{2t}}{2} = \frac{pr}{4t} = \frac{\sigma_\theta}{4}$$

4 $$e_z = \frac{1}{E}\left[\sigma_z - \nu\left(\cancel{\sigma_r}^{\,0} + \sigma_\theta\right)\right] = \frac{1}{E}\left(\frac{pr}{2t} - \nu\frac{pr}{t}\right) \equiv e_0$$

$$\therefore \quad \frac{1-2\nu}{2E}\frac{pr}{t} = e_0 \Rightarrow \boxed{p = \frac{2E}{1-2\nu}\frac{t}{r}e_0}$$

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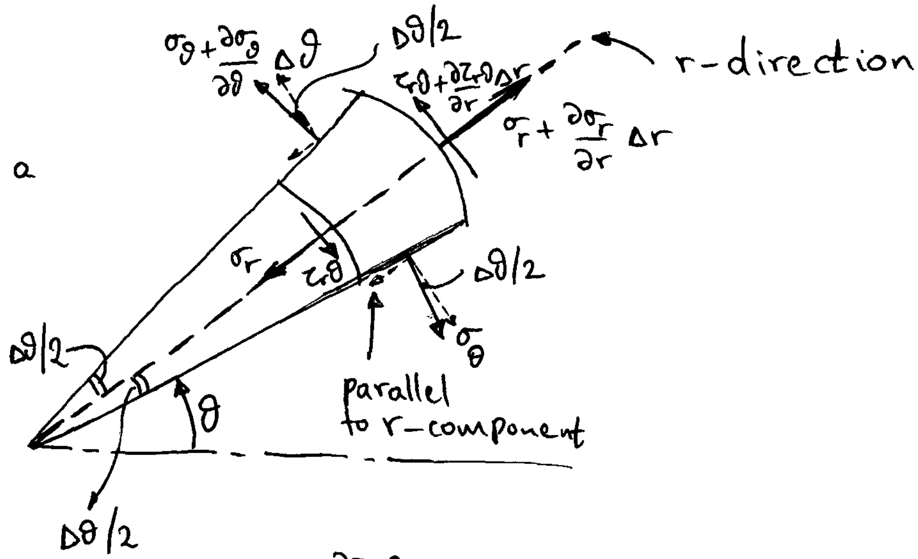
Fig a

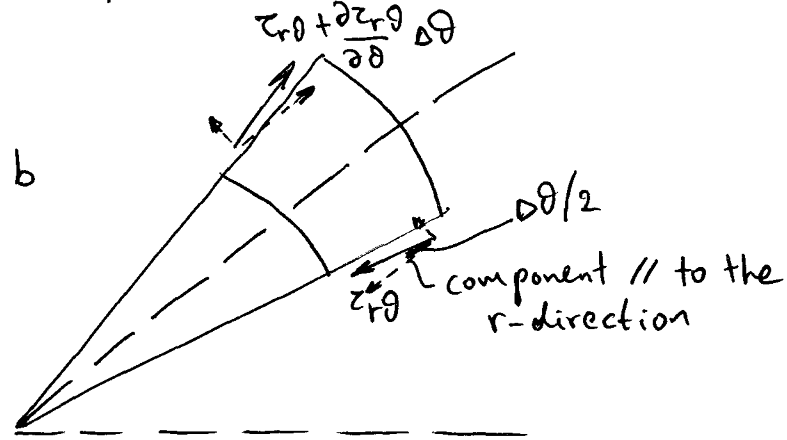


Note:

$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$

$\cos\frac{\Delta\theta}{2} \approx 1$

Fig b



For clarity $\tau_{r\theta}$ is shown in this Figure

$\sum F_r = 0 \implies$

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r}\Delta r\right)\underbrace{\left[(r+\Delta r)\Delta\theta\right]}_{\text{area}} - \sigma_r\,[r\Delta\theta]$$

$$-\sigma_\theta\underbrace{[\Delta r]}_{\text{area}}\underbrace{\frac{\Delta\theta}{2}}_{\text{r-component}} - \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}\Delta\theta\right)[\Delta r]\frac{\Delta\theta}{2}$$

$$-\tau_{r\theta}\,[\Delta r]\underbrace{1}_{\text{note } \cos\frac{\Delta\theta}{2}\simeq 1} + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}\Delta\theta\right)[\Delta r]\,1 = 0 \implies$$

$$\sigma_r \, \Delta r \, \Delta\theta + \frac{\partial \sigma_r}{\partial r} \Delta r \, r \, \Delta\theta + \underbrace{\cancel{\frac{\partial \sigma_r}{\partial r} (\Delta r)^2 (\Delta\theta)}}_{\text{neglect (small)}}$$

$$- \sigma_\theta \frac{\Delta\theta}{2} \Delta r - \sigma_\theta \frac{\Delta\theta}{2} \Delta r - \cancel{\frac{\partial \sigma_\theta}{\partial \theta} (\Delta\theta)^2 \frac{\Delta r}{2}}$$

$$+ \frac{\partial \tau_{r\theta}}{\partial \theta} \Delta r \, \Delta\theta = 0 \Longrightarrow \text{(Divide by } r \Delta r \Delta\theta)$$

$$\boxed{\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} = 0}$$

Similarly, $\Sigma F_\theta = 0 \Longrightarrow$

$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} \Delta\theta\right) \underbrace{\Delta r}_{\text{area}} \underbrace{1}_{\cos \Delta\theta/2} - \sigma_\theta \, \Delta r \, 1$$

$$+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} \Delta r\right) (r + \Delta r) \Delta\theta - \tau_{r\theta} \, r \, \Delta\theta \quad \text{(from Fig a)}$$

$$+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} \Delta\theta\right) \Delta r \frac{\Delta\theta}{2} + \tau_{r\theta} \frac{\Delta\theta}{2} \Delta r = 0 \Longrightarrow$$

$$\frac{\partial \sigma_\theta}{\partial \theta} \Delta r \, \Delta\theta + \frac{\partial \tau_{r\theta}}{\partial r} r \, \Delta r \, \Delta\theta + \underbrace{\cancel{\frac{\partial \tau_{r\theta}}{\partial r} (\Delta r)^2 \Delta\theta}}_{\text{neglect}} + \tau_{r\theta} \, \Delta r \, \Delta\theta +$$

$$+ \tau_{r\theta} \, \Delta r \, \Delta\theta + \cancel{\frac{\partial \tau_{r\theta}}{\partial \theta} (\Delta\theta)^2 \frac{\Delta r}{2}} \Longrightarrow$$

$$\boxed{\frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2 \frac{\tau_{r\theta}}{r} = 0}$$

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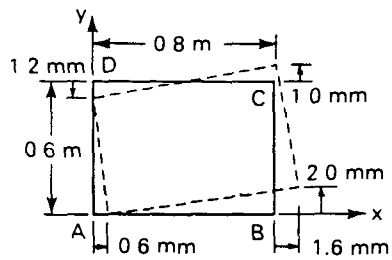

The following approximate version of the strain-displacement relations given in the notes must be used:

$$e_x = \frac{\Delta u}{\Delta x}, \quad e_y = \frac{\Delta v}{\Delta y}, \quad \gamma_{xy} = \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}$$

Thus, by setting $\Delta x = 800$ mm, $\Delta y = 600$ mm, the normal strains are calculated as follows:

$$e_x = \frac{u_B - u_A}{\Delta x} = \frac{1.6 - 0.6}{800} = 1250\ \mu$$

$$e_y = \frac{U_D - U_A}{\Delta y} = \frac{-1.2 - 0}{600} = -2000\ \mu$$

In a like manner, we obtain the shearing strain:

$$\gamma_{xy} = \frac{u_D - u_A}{\Delta y} + \frac{U_B - U_A}{\Delta x} = \frac{0 - 0.6}{600} + \frac{2 - 0}{800} = 1500\ \mu$$

The positive sign indicates that angle BAD has decreased.